SUPERCOMPACTNESS WITHIN THE PROJECTIVE HIERARCHY

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Abstract. We show that all the projective ordinals $\boldsymbol{\delta}^1_n$ are supercompact through their supremum $\aleph_{\epsilon_0}$, and a ways beyond.

1. Introduction

We work throughout in the theory $ZF + DC + AD$. The main goal of this paper is to show the following theorem (the definitions are given below).

Theorem 1.1. *Every projective ordinal* $\boldsymbol{\delta}^1_n$ *is* $(\sup_n \boldsymbol{\delta}^1_n)$*-supercompact.*

In fact, we will show that the $\boldsymbol{\delta}^1_n$ are λ-supercompact for even larger λ.

We remark that our assumption of DC is mainly for convenience, and the careful reader can check that the proof of theorem 1.1 (and theorem 4.1) goes through under just $ZF + AD$. This is because all uses of DC in the proof can be reduced to an instance of $DC_{\mathbb{R}}$ (the axiom that if $A \subseteq \mathbb{R} \times \mathbb{R}$ is such that $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ A(x, y)$, then $\exists f \colon \omega \to \mathbb{R}\ \forall n\ A(f(n), f(n+1)))$ where A is Suslin, whence the existence of f is automatic.

We refer the reader to [9] or [12] for the basic results in determinacy theory we use, such as the coding lemma. Recall the projective ordinals, $\boldsymbol{\delta}^1_n$, $n \in \omega$, are defined by: $\boldsymbol{\delta}^1_n$ is the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of ω^ω (which, it turns out, is the same as the supremum of the lengths of the $\boldsymbol{\Sigma}^1_n$ well-founded relations on ω^ω). These ordinals play an important role in the descriptive set theory of the projective sets; we refer the reader to [12] for this development.

Recall a set $A \subseteq \omega^\omega$ is said to be κ-Suslin if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $A = p[T]$, that is, $x \in A \leftrightarrow \exists f \in \kappa^\omega\ \forall n\ (x \restriction n, f \restriction n) \in T$. A cardinal κ is said to be a *Suslin cardinal* if there is a set $P \subseteq \omega^\omega$ which is κ-Suslin but not κ'-Suslin for any $\kappa' < \kappa$. An ordinal δ is said to be *reliable* if there is a set $P \subseteq \omega^\omega$ and a scale $\{\phi_n\}_{n\in\omega}$ on P with each $\phi_n \colon P \to \delta$ and with $\phi_0 \colon P \overset{onto}{\to} \delta$. In the Kechris-Woodin theory of generic codes (*c.f.* [10]; we review later what we need from this theory), they adopt a slight technical strengthening of this definition by requiring also that the scale relations

$$x \leq_n y \Leftrightarrow x, y \in P \wedge \phi_n(x) \leq \phi_n(y)$$
$$x <_n y \Leftrightarrow x, y \in P \wedge \phi_n(x) < \phi_n(y)$$

be Suslin. This extra assumption, for example, is automatically satisfied if $V = L(\mathbb{R})$. At any rate, we will be using reliability only within the projective ordinals in this paper, so the distinction is immaterial and we officially adopt this stronger definition. Reliable ordinals are important because (among other reasons) of their role in the generic coding theory (*c.f.* theorem 3.1). Every Suslin cardinal is reliable, and assuming $V = L(\mathbb{R})$ it is shown in [15] that every reliable cardinal in Suslin, a result attributable to some combination of Jackson, Kechris, Martin and Steel. There are, however, many more reliable ordinals, as they for example are c.u.b. in every regular Suslin cardinal. All we require in this regard for this paper is that the $\boldsymbol{\delta}^1_{2n+1}$ are reliable (in fact are Suslin cardinals), an immediate consequence of the scale property for $\boldsymbol{\Pi}^1_{2n+1}$ (*c.f.* [12]).

If δ is reliable and $S \subseteq \delta$, recall S is said to be *honest* if for all $\alpha \in S$ there is an $x \in \omega^\omega$ such that $\phi_0(x) = \alpha$ and $\forall n\ \phi_n(x) \in S$. This notion is, of course, defined relative to P, $\{\phi_n\}$.

It is a classical result that $\boldsymbol{\delta}^1_1 = \aleph_1$. Martin showed that $\boldsymbol{\delta}^1_2 = \aleph_2$, and $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, and Martin and Kunen (independently) showed $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$. In general, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ (Kunen, Martin), and $\boldsymbol{\delta}^1_{2n+1}$ is the successor of a cardinal of cofinality ω (Kechris). The Suslin cardinals below the projective ordinals are precisely the $\boldsymbol{\delta}^1_{2n+1}$ and their predecessors $(\boldsymbol{\delta}^1_{2n+1})^-$. These results may be found in [9]. Later, through a more detailed analysis, Jackson computed explicitly $\boldsymbol{\delta}^1_{2n+3} = \aleph_{\omega(2n+1)+1}$, where $\omega(0) = 1$, and $\omega(n+1) = \omega^{\omega(n)}$ (ordinal exponentiation). In [4] this analysis is given for the case $\boldsymbol{\delta}^1_5$.

Work of Moschovakis, Kunen, Martin, and Solovay showed that all of the $\boldsymbol{\delta}^1_n$ are regular, and in fact measurable (that is, κ carries a κ-complete, normal, ultrafilter). We refer the reader to [12] for a more detailed history. These results are all of a "local" nature for $\boldsymbol{\delta}^1_n$, that is, they do not involve considering ordinals greater than $\boldsymbol{\delta}^1_n$. Supercompactness is different in this respect. Recall that an ultrafilter ν on $\mathcal{P}_\kappa(\lambda)$ is *fine* if for all $A \in \mathcal{P}_\kappa(\lambda)$, $\forall^*_\nu S\ A \subseteq S$ (here $\forall^*_\nu S$ denotes "for almost S with respect to the measure ν"). ν is *normal* if whenever $F \colon \mathcal{P}_\kappa(\lambda) \to \mathrm{ON}$ is pressing down almost everywhere, that is, $\forall^*_\nu S\ F(S) \in S$, then F is constant almost everywhere.

Definition 1.1. κ is λ-supercompact, where $\kappa \leq \lambda \in \mathrm{ON}$, if there is a fine, normal, κ-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

It is easy to see that κ is κ-supercompact iff κ is measurable, so supercompactness provides a strengthening of the notion of measurability. Also, if κ is λ-supercompact, then easily κ is λ' supercompact for all $\kappa \leq \lambda' < \lambda$.

Solovay showed, assuming $AD^{\mathbb{R}}$, that ω_1 is λ-supercompact for all $\lambda < \Theta$. Harrington and Kechris [3] proved an ordinal determinacy result which showed, when combined with the later Martin-Steel result that $(\boldsymbol{\Sigma}^2_1)^{L(\mathbb{R})}$ has the scale property assuming AD, that ω_1 is $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$-supercompact assuming just AD. Becker [1], using different techniques, showed that $\boldsymbol{\delta}^1_2 = \omega_2$ is $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$-supercompact assuming AD. The techniques used in that proof, however, did not immediately generalize to the higher $\boldsymbol{\delta}^1_n$ (and it is still not clear if they can be made to do so).

The question we consider here is for which λ can we show that the $\boldsymbol{\delta}^1_n$ are λ-supercompact? Becker initiated this investigation by formulating the notion of κ being λ-regular, and used it to prove theorem 2.1 below. Jackson then showed that all the $\boldsymbol{\delta}^1_n$ are λ-regular for λ below the common supremum of the $\boldsymbol{\delta}^1_n$, namely $\lambda = \aleph_{\epsilon(0)}$, where $\epsilon(0) = \sup_n \omega(n)$. It follows that the $\boldsymbol{\delta}^1_n$ are all $\lambda = \aleph_{\epsilon(0)}$ supercompact. In fact, the same proof shows that the $\boldsymbol{\delta}^1_n$ are $\aleph_{\aleph_1}$-supercompact. We present these results in §2, §4. This was subsequently extended to somewhat larger values of λ using more of the (generalized) projective hierarchy analysis.

More recently, Jackson formulated and established a certain combinatorial property, assuming $AD + V = L(\mathbb{R})$. This "weak square" property, denoted $\boxminus_{\kappa,\lambda}$, was shown to hold for all Suslin cardinals κ, and all $\lambda < \Theta$. Using these results, Jackson showed, assuming $AD + V = L(\mathbb{R})$, that every regular κ which is either a Suslin cardinal or the successor of a Suslin cardinal is $\boldsymbol{\delta}^2_1$-supercompact. In particular, all the $\boldsymbol{\delta}^1_n$ are $\boldsymbol{\delta}^2_1$-supercompact (as noted in remark 3.1 of [6], assuming just AD those results still give that the $\boldsymbol{\delta}^1_n$ are $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$-supercompact). These results appear separately in [6]. Although superseded by these more recent results, we nevertheless feel the methods of this paper are of independent interest. In particular, the methods used in the proof of theorem 2.1 may prove useful elsewhere.

2. λ-regular cardinals

In this section we formulate the notion of a cardinal κ being λ-regular and prove a theorem which connects this with the λ-supercompactness of κ. Both the definition and theorem are due to Becker.

Definition 2.1. Let κ be a cardinal and $\lambda \geq \kappa$. We say κ is λ-regular if for any $\alpha < \kappa$ and function $f\colon \alpha \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$, there is an $A \in \mathcal{P}_\kappa(\lambda)$ such that $\forall \beta < \alpha\ A \supseteq B$ for some $B \in f(\beta)$.

The λ-regularity of κ is a choice-like property, and follows trivially from AC for regular κ (which is of no use to us here). We note the following simple facts.

(a) κ is κ regular iff κ is a regular cardinal.
(b) κ is λ-regular $\Rightarrow$ κ is λ'-regular for all $\kappa \leq \lambda' \leq \lambda$.
(c) If κ is a regular cardinal and $\lambda \geq \kappa$, then κ^+ is λ-regular $\Rightarrow$ κ is λ-regular.

Proof. Suppose $f\colon \alpha \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$, where $\alpha < \kappa$. Since κ^+ is λ-regular, let $C \subseteq \lambda$, $|C| \leq \kappa$, "cover" f, that is, $\forall \beta < \alpha\ \exists B \in f(\beta)\ B \subseteq C$. As κ is regular, some $A \subseteq C$ with $|A| < \kappa$ also covers f. $\square$

(d) If κ is λ'-regular for all $\lambda' < \lambda$, and $\mathrm{cof}(\lambda) \geq \kappa$, then κ is λ-regular.
(e) $\forall \lambda\ \aleph_1$ is λ-regular (by countable choice).
(f) κ is λ-supercompact $\Rightarrow$ κ is λ-regular.

Remark 2.1. A generic coding argument (using just generic codes for countable ordinals) also shows:

(g) $\forall \lambda < \Theta$, $\aleph_2$ is λ-regular.

This, and theorem 2.1 below provide another proof of Becker's result that $\aleph_2$ is $\boldsymbol{\delta}^2_1$-supercompact.

Proof of (g) (sketch). Let $\lambda < \Theta$, and $f\colon \aleph_1 \to \mathcal{P}(\mathcal{P}_{\aleph_2}(\lambda)) - \{\emptyset\}$. Let $WO \subseteq \omega^\omega$ denote the usual set of codes for countable ordinals. By the coding lemma, there is a function $F\colon WO \to \mathcal{P}_{\aleph_2}(\lambda)$ such that $\forall x \in WO\ F(x) \in f(|x|)$, where $|x| < \aleph_1$ is the ordinal coded by x. Let $G\colon (\omega_1)^\omega \to WO \subseteq \omega^\omega$ be a (Lipschitz) continuous function such that if $\vec{\alpha} = (\alpha(0), \alpha(1), \ldots)$ then $|G(\vec{\alpha})| \leq \omega + \sup_n \alpha(n)$, and if $\vec{\alpha}$ enumerates a countable ordinal $\gamma \geq \omega^2$, then $|G(\vec{\alpha})| = \gamma$. [As $\alpha(0), \alpha(1), \ldots$ are given, G builds the well-ordering $\prec$ of ω such that $n \prec m$ iff $\alpha(n) < \alpha(m)$ for n, m *new*, where we say i is new if $\alpha(i) \neq \alpha(j)$ for all $j < i$. The non-new integers are ordered in their natural order and $\prec$-precede all the new integers. G is a "generic coding function" for the countable ordinals.] Let $\forall^* s \in \alpha^\omega$ abbreviate "for comeager many $s \in \alpha^\omega$," where α^ω carries the product of the discrete topology on α (so α^ω is homeomorphic to ω^ω). Likewise, if $p \in \alpha^{<\omega}$, let $\forall^*_p s \in \alpha^\omega$ abbreviate "for comeager in the neighborhood N_p many $s \in \omega^\omega$" where N_p is the set of all $s \in \alpha^\omega$ extending p. For convenience, assume $\mathrm{dom}(f) = \aleph_1 - \omega^2$. For $\alpha \in \mathrm{dom}(f)$, define $A_\alpha \subseteq \lambda$ by: $A_\alpha = \bigcup_{p \in \alpha^{<\omega}} A_\alpha(p)$, where $\gamma \in A_\alpha(p)$ iff $\forall^*_p s \in \alpha^\omega$ $[\gamma \in F(G(s))]$. Using the full additivity of category on α^ω (a consequence of AD), it is easy to see that each $A_\alpha(p)$, and hence each A_α, has size $\leq \aleph_1$. Let $A = \bigcup_\alpha A_\alpha \in \mathcal{P}_{\aleph_2}(\lambda)$. By additivity again, and the fact that all sets have the Baire property, it follows that $\forall \alpha \in \mathrm{dom}(f)\ \forall^* s \in \alpha^\omega\ F(G(s)) \subseteq A_\alpha \subseteq A$, so A covers f as desired. $\square$

The following theorem is the main result of this section, and shows the significance of the notion of λ-regular.

Theorem 2.1. *Suppose $\kappa \leq \lambda$, κ is either a Suslin cardinal or the successor of a Suslin cardinal, and λ is less than the supremum of the Suslin cardinals. If κ is λ-regular, then κ is λ-supercompact.*

Fix $\kappa \leq \lambda$ as in the statement of the theorem, so κ is regular. Let $\preceq$ be a prewellordering of ω^ω length $\geq \lambda$, and fix a pointclass $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$ with $\preceq \in \mathbf{\Gamma}$. Let $|y|$ denote the rank of y in the prewellordering $\preceq$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\mathbf{\Gamma}$ set. For $x \in \omega^\omega$, let $S_x = \{\alpha < \lambda \colon \exists y\ [U(x,y) \wedge |y| = \alpha]\}$ if this set has size $< \kappa$, and otherwise let $S_x = \emptyset$. By the coding lemma, $\forall S \in \mathcal{P}_\kappa(\lambda)\ \exists x\ S_x = S$. Thus, the map $x \to S_x$ provides a coding for $\mathcal{P}_\kappa(\lambda)$.

Suppose now that $A \subseteq \mathcal{P}_\kappa(\lambda)$, and consider the following real game G_A:

I	x_0		x_2	$\cdots$	
II		x_1		x_3	$\cdots$

Here I and II alternate playing $x_i \in \omega^\omega$. Each player must play x_i such that $\bigcup_{j<i} S_{x_j} \subseteq S_{x_i}$; the first player to violate this condition loses. Otherwise, we define II to win the run of G_A iff $\bigcup_i S_{x_i} \in A$. Although this is a real game, it is determined assuming just AD. This follows from the arguments of §6 of [2], the fact that there is a Suslin cardinal $\geq \lambda$, and that the payoff of the game depends only on the set $\bigcup_i S_{x_i}$. To be precise, we abstract the relevant determinacy arguments of [2] in the following theorem.

Remark 2.2. For the reader willing to strengthen our determinacy assumptions to $AD^{\mathbb{R}}$, the determinacy of the game G_A is immediate.

Definition 2.2. We say a boldface pointclass $\mathbf{\Delta}$ is reasonable if it satisfies:

(1) $\mathbf{\Delta}$ is closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$, and complements.
(2) Every set $A \subseteq \omega^\omega \times \omega^\omega$ in $\mathbf{\Delta}$ has a $\mathbf{\Delta}$ uniformization.

Theorem 2.2 (AD)**.** *Suppose $\phi_1 \colon P_1 \to \kappa_1$, $\phi_2 \colon P_2 \to \kappa_2$ are norms with corresponding prewellorderings $\preceq_1, \preceq_2$. Suppose $P_1, P_2, \preceq_1, \preceq_2 \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ is reasonable. Suppose $\kappa \leq \kappa_2$ is regular, and $A \subseteq \kappa_1^\omega \times \mathcal{P}_\kappa(\kappa_2)$. Consider the following real game G_A:*

I	x_0, y_0		x_2, y_2	$\cdots$	
II		x_1, y_1		x_3, y_3	$\cdots$

The players take turns playing reals x_i, y_i. The first player to violate one of the following rules loses. Each x_i must be in P_1, and hence codes an ordinal $\phi_1(x_i)$.

Each y_i must code a set $S_{y_i} \in \mathcal{P}_\kappa(\kappa_2)$. The coding here is with respect to some (fixed) non self-dual pointclass $\mathbf{\Gamma}_2 \subseteq \mathbf{\Delta}$ which contains $P_2, \preceq_2$ and is closed under $\exists^{\omega^\omega}$, countable unions and intersections (which we can always find by the closure of $\mathbf{\Delta}$). That is, if $U \subseteq \omega^\omega \times \omega^\omega$ is universal for $\mathbf{\Gamma}_2$, then we require that $U_{y_i} \doteq \{z \colon U(y_i, z)\} \subseteq P_2$ and $\{\phi_2(z) \colon z \in U_{y_i}\}$ has cardinality less than κ. Finally, we require that $S_{y_i} \supseteq \bigcup_{j<i} S_{y_j}$. If both players satisfy the rules, then II wins iff $A(\vec{s}, S)$, where $\vec{s} = (\phi_1(x_0), \phi_1(x_1), \dots)$, and $S = \bigcup_i S_{y_i}$.

Then the game G_A is determined, and moreover, the player with a winning strategy has a winning strategy all of whose "levels" lie in $\mathbf{\Delta}$ (by level we mean the strategy restricted to positions of a fixed length).

Proof. We sketch the proof for the reader familiar with [2]. We fix a sequence of Spector pointclasses $\mathbf{\Gamma}_0^{\mathrm{I}} \subsetneq \mathbf{\Gamma}_0^{\mathrm{II}} \subsetneq \mathbf{\Gamma}_1^{\mathrm{I}} \subsetneq \mathbf{\Gamma}_1^{\mathrm{II}} \cdots$ within $\mathbf{\Delta}$, all closed under $\forall^{\omega^\omega}$ and such that any $\mathbf{\Gamma}_i^{\mathrm{I}}$ (resp. $\mathbf{\Gamma}_i^{\mathrm{II}}$) relation has (uniformly) a uniformization in $\mathbf{\Delta}_i^{\mathrm{II}}$ (resp. $\mathbf{\Delta}_{i+1}^{\mathrm{I}}$). Assume also that P_1, P_2, $\preceq_1$, $\preceq_2$, $\mathbf{\Gamma}_2$ are all in $\mathbf{\Delta}_0^{\mathrm{I}}$. As in [2], we consider the integer Harrington-Kechris type game G_A^* where (using the notation of [2]) I plays out $(a_0^{\mathrm{I}}, c_0^{\mathrm{I}})$, and II plays out $(a_0^{\mathrm{II}}, c_0^{\mathrm{II}})$. In the usual Harrington-Kechris manner, I's play determines a sequence a_0^{I}, c_0^{I}, d_0^{I}, a_1^{I} c_1^{I}, d_1^{I}, …, where each $a_i^{\mathrm{I}} = \langle e_i, b_i\rangle$ ($e_i \in \omega$) attempts to code a perfect set relative to $\mathbf{\Gamma}_i^{\mathrm{I}}$ (coded as a tree $T_{a_i^{\mathrm{I}}}$ on $\{0,1\}$, the characteristic function of this tree is the $\Gamma_i^{\mathrm{I}}(b_i)$ relation with code e_i). Also, $c_i^{\mathrm{I}} \in [T_{a_i^{\mathrm{I}}}]$, and $\langle d_i^{\mathrm{I}}, a_{i+1}^{\mathrm{I}}, c_{i+1}^{\mathrm{I}}\rangle$ is the image of c_i^{I} under the natural homeomorphism $\pi_{a_i^{\mathrm{I}}}$ of $[T_{a_i^{\mathrm{I}}}]$ with 2^ω. Each d_i^{I} codes two reals x_i, y_i, where x_i codes an ordinal $< \kappa_1$ and y_i codes an element S_{y_i} of $\mathcal{P}_\kappa(\kappa_2)$. Likewise for II (the first player failing to meet these requirements loses). If all of the x_i, y_i are defined, then we have defined a run of G_A, and whichever player wins that run of G_A wins this run of G_A^*. As in the proof of lemma B of [2], we can show that the player with a winning strategy in G_A^* has one for G_A as well. We summarize the main points.

First, the hypotheses on our pointclasses are enough to define a strategy in the game G_A as in [2]. Suppose for example t is a winning strategy for II in G_A^*, and we consider the first level of II's strategy for G_A. Following [2], and using that notation, fix a recursive a_0^{I} coding a perfect set (relative to $\mathbf{\Gamma}_0^{\mathrm{I}}$), and suppose I plays (x_0, y_0) in G_A (which we assume is not an immediate loss). Let $d_0^{\mathrm{I}} = \langle x_0, y_0\rangle$. For any $(a,c) \in (2^\omega)^2$, let $c_0^{\mathrm{I}}(a,c)$ be the real such that $\pi_{a_0^{\mathrm{I}}}(c_0^{\mathrm{I}}(a,c)) = \langle d_0^{\mathrm{I}}, a, c\rangle$. Let $d_0^{\mathrm{II}}(a,c)$ be the d_0^{II} produced following t against $(a_0^{\mathrm{I}}, c_0^{\mathrm{I}}(a,c))$. Let $d_0^{\mathrm{II}}(a,c) = \langle x_1(a,c), y_1(a,c)\rangle$. Fix $b \equiv_T \langle t, d_0^{\mathrm{I}}\rangle$. For $e \in \omega$ and $c \in 2^\omega$, let $d_0^{\mathrm{II}}(e,c) = \langle x_1(e,c), y_1(e,c)\rangle$ be from II's response following t against I's play of a_0^{I} and $c_0^{\mathrm{I}}(a,c)$, where $a = \langle e, b\rangle$. Let $\delta^e(c) < \kappa_1$ be the ordinal coded by $x_1(e,c)$, and $D^e(c) \in \mathcal{P}_\kappa(\kappa_2)$ the set coded by $y_1(e,c)$. A perfect binary tree $T \subseteq 2^{<\omega}$ is said to be e-good if

(1) $\forall c, c' \in [T]\ \delta^e(c) = \delta^e(c')$.
(2) $\forall c \in [T]\ \forall \psi < \kappa_2\ \exists p \prec c\ \forall c'\ [(p \prec c' \wedge c' \in [T]) \implies (\psi \in D^e(c) \leftrightarrow \psi \in D^e(c'))]$.

The usual category arguments show that for each e there is an e-good tree. Moreover, the relation $R(e,T) \leftrightarrow (T \text{ is } e\text{-good})$ is $\Gamma_0^{\mathrm{II}}(b)$. Since (uniformly) $\mathbf{\Gamma}_0^{\mathrm{II}}$ relations have $\mathbf{\Delta}_1^{\mathrm{I}}$ uniformizations, there is a recursive function $e \to \hat{e}$ such that for all e, $\langle \hat{e}, b\rangle$ codes a perfect tree $T_{\langle \hat{e}, b\rangle}$ (with respect to $\mathbf{\Gamma}_1^{\mathrm{I}}$) which is e-good. As usual, find e_0 by the recursion theorem such that $T_{\langle e,b\rangle} = T_{\langle \hat{e}, b\rangle}$. Let $a_1^{\mathrm{I}} = \langle e_0, b\rangle$. II's response to (x_0, y_0) will be some (x_1, y_1) such that $x_1 \in P_1$ and x_1 codes the stable value of $\delta^{e_0}(c)$ for $c \in [T_{a_1^{\mathrm{I}}}]$, and y_1 codes the union of the $D^{e_0}(c)$ for $c \in [T_{a_1^{\mathrm{I}}}]$. Applying uniformization, we may assume the map $(x_0, y_0) \to (x_1, y_1)$ is in $\mathbf{\Delta}$, and this constitutes the first level of II's strategy τ in G_A^*. The other levels are defined similarly and are also in $\mathbf{\Delta}$.

It follows from (2) that the y_1 of II's response codes a set of size $< \kappa$. It follows that τ does not lose trivially for II. Let $(x_0, y_0), (x_1, y_1), \dots$ be a run of τ. Let $(a_0^{\mathrm{I}}, c_0^{\mathrm{I}})$ be the reals which "decode" into $d_i^{\mathrm{I}} = (x_{2i}, y_{2i})$ and the a_i^{I} as defined above. Let $d_i^{\mathrm{II}} = \langle x'_{2i+1}, y'_{2i+1} \rangle$ be the reals produced by II following t. Clearly $\phi_1(x'_{2i+1}) = \phi_1(x_{2i+1})$, and $S_{y'_{2i+1}} \subseteq S_{y_{2i+1}}$. However, the increasing condition on the game shows that in both games the sets of ordinals played have the same union, namely $\bigcup_i S_{y_{2i}}$. Since t was winning for II in G_A^*, τ is also winning for II in G_A. □

We define now $\mu \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))$ by: $A \in \mu$ iff II has a winning strategy in G_A. It is easy to see that μ is a countably additive, fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$. We write $\mu(A) = 1$ for $A \in \mu$ from now on. We must show that μ is κ-complete and normal. The following lemma shows that for measures μ defined via games in this manner, normality follows from κ-completeness.

Lemma 2.1. *If μ is κ-additive, then μ is normal.*

Proof. Assume μ is κ-additive, and let $F \colon \mathcal{P}_\kappa(\lambda) \to \mathrm{ON}$ with $F(S) \in S$ for all S. Towards a contradiction, assume there is no μ measure one set on which F is constant. Using DC, we define a run of the game using a standard dove-tailing argument as follows. Let $n, m \to \langle n, m \rangle$ be a bijection between ω^2 and ω which is increasing in each argument (in particular, $\langle 0, 0 \rangle = 0$), and let $i \to ((i)_0, (i)_1)$ be its inverse. Let x_0 code an arbitrary $S_{x_0} \in \mathcal{P}_\kappa(\lambda)$, and for $n > 0$ we let $x_{2n} = x_{2n-1}$, *i.e.* I will copy II's moves. Let σ_0 be a winning strategy for II to produce an S such that $F(S) \notin S_{x_0}$, which exists by κ-additivity. Let $x_1 = \sigma_0(x_0)$. In general, suppose $x_0, \dots, x_{2n}$ have been defined and code sets $S_{x_0} \subseteq \cdots \subseteq S_{x_{2n}} \in \mathcal{P}_\kappa(\lambda)$. Suppose also strategies $\sigma_0, \sigma_2, \dots, \sigma_{2n}$ for II have been fixed so that if S results from a run according to σ_{2i}, then $F(S) \notin S_{x_{2i}}$. Let $a = (n)_0$, $b = (n)_1$. Let x_{2n+1} be the response of σ_a against

$$(x_{2\langle a,0\rangle}, x_{2\langle a,0\rangle+1}, x_{2\langle a,1\rangle}, x_{2\langle a,1\rangle+1}, \dots, x_{2\langle a,b\rangle} = x_{2n}).$$

Fix then σ_{2n+2} winning for II to produce S such that $F(S) \notin S_{x_{2n+2}} = S_{x_{2n+1}}$. Continuing, we define x_i for all i. This, however, is a contradiction, since $S \doteq \bigcup_i S_{x_i} \in \mathcal{P}_\kappa(\lambda)$ and $F(S) \notin S$. □

Suppose towards a contradiction that μ is not κ-additive, and let $\alpha < \kappa$ be least such that there is a sequence $\{A_\beta\}_{\beta<\alpha}$ with $\mu(A_\beta) = 0$ for all β, but $\mu(\bigcup_{\beta<\alpha} A_\beta) = 1$. Fix such a sequence A_β, and without loss of generality suppose it is increasing and $\bigcup_{\beta<\alpha} A_\beta = \mathcal{P}_\kappa(\lambda)$. Note that α is regular.

If κ is a regular Suslin cardinal, then there is an α' with $\alpha \leq \alpha' < \kappa$ such that α' is reliable. If κ is the successor of a Suslin cardinal, say $\kappa = (\alpha')^+$, then α' is reliable and $\alpha \leq \alpha'$ (as α is a cardinal). Thus, in either case there is a reliable ordinal α', which we now fix, such that $\alpha \leq \alpha' < \kappa$.

Let $P \subseteq \omega^\omega$, and $\{\phi_n\}_{n\in\omega}$ be a regular scale on P whose 0^{th} norm ϕ_0 maps P onto α, and with all norms $\phi_n \colon P \to \alpha'$. Let $T \subseteq (\omega \times \alpha')^{<\omega}$ be the tree of the scale $\{\phi_n\}$, so $P = p[T]$. For $\beta < \alpha$, let $T_\beta = \{(y(0), \dots, y(n)), (\beta(0), \dots, \beta(n)) \in T \colon \beta(0) \leq \beta\}$. Consider the following real game G:

$$\begin{array}{lcccccc} \mathrm{I} & x_0 & & x_2 & & \cdots & \\ \mathrm{II} & & x_1 & & x_3 & & \cdots \\ & & y(0) & & y(1) & & \cdots \end{array}$$

Here I and II play reals x_i, and II plays also integers $y(i)$, building a real $y \in \omega^\omega$. Each player must play a real x_i coding an element of $\mathcal{P}_\kappa(\lambda)$ and such that $\bigcup_{j<i} S_{x_j} \subseteq S_{x_i}$; the first player who fails to do so loses. Otherwise, II wins iff $y \in P$ and $S \in A_{\phi_0(y)}$, where $S = \bigcup_i S_{x_i}$. By theorem 2.2 this game is determined assuming just AD. Now, I cannot win this game, for if σ were winning for I, we define a sequence $x_1, x_3, \ldots$ of reals coding $S_{x_1} \subseteq S_{x_3} \subseteq \cdots$ such that for any $y \in \omega^\omega$, if $x_0, x_2, \ldots$ is the result of following σ against II's play of $(x_1, y(0)), (x_3, y(1)), \ldots$, then $S_{x_0} \subseteq S_{x_1} \subseteq S_{x_2} \subseteq \cdots$. Let then β be such that $S = \bigcup_i S_{x_i} \in A_\beta$, and fix $y \in P$ with $\phi_0(y) = \beta$. Playing $x_1, x_3, \ldots$ and y then defeats σ, a contradiction. Fix a winning strategy τ for II in G. Let $\tau_0(x_0, x_2, \ldots, x_{2n})$ denote the real part of τ's last move against I's play of $(x_0, x_2, \ldots, x_{2n})$, and let $\tau_1(x_0, x_2, \ldots, x_{2n})$ be the integer part.

If $\vec{z} = z_0, z_1, \ldots$ is a finite or infinite sequence of reals each coding $S_{z_i} \in \mathcal{P}_\kappa(\lambda)$, let $\tau_1 * \vec{z}$ be the (finite or infinite) sequence of integers $\tau_1(z_0)$, $\tau_1(z_0, z_1), \ldots$.

Definition 2.3. We say a finite or infinite sequence $z_0, z_1, \ldots$ is *good* if for all $i \geq 0$, $S_{z_i} \cup S_{\tau_0(z_0,\ldots,z_i)} \subseteq S_{z_{i+1}}$.

Note that if $\vec{z}$ is an infinite good sequence, then $y = \tau_1 * \vec{z} \in P$ and $S \in A_{\phi_0(y)}$, where $S = \bigcup_i S_{z_i}$.

Define a tree V on $\omega^\omega \times \omega \times \mathrm{ON}$ by:

$$\vec{s} = ((z_0, y(0), \beta(0)), (z_1, y(1), \beta(1)), \ldots, (z_n, y(n), \beta(n))) \in V$$

iff the following hold:

(1) $(z_0, z_1, \ldots, z_n)$ is good.
(2) $(y(0), \ldots, y(n)) = \tau_1 * (z_0, \ldots, z_n)$.
(3) $((y(0), \ldots, y(n)), (\beta(0), \ldots, \beta(n)) \in T$.

For $\beta < \alpha$, let $V_\beta \subseteq V$ be defined just as V except we also require that $\beta(0) \leq \beta$. Note that if $((z_0, y(0), \beta(0)), (z_1, y(1), \beta(1)), \ldots$ is a path through V_β, then $y \in P$, $\phi_0(y) \leq \beta$, and by goodness, $S \in A_{\phi_0(y)} \subseteq A_\beta$, where $S = \bigcup_i S_{z_i}$.

We perform a Cantor-Bendixson type analysis on V. If W is a subset of V, we define its derivative $W' \subseteq W$ by: $\vec{s} \in W'$ (in the above notation for $\vec{s}$) iff $\vec{s} \in W$ and for every good $(z_0, \ldots, z_n, z'_{n+1})$ extending $(z_0, \ldots, z_n)$, there is a z_{n+1} with $S_{z_{n+1}} \supseteq S_{z'_{n+1}}$ and $y(n+1)$, $\beta(n+1)$ such that $\vec{s}^\frown(z_{n+1}, y(n+1), \beta(n+1)) \in W$. Define $W^0 = W$, $W^{\gamma+1} = (W^\gamma)'$, and for limit γ, $W^\gamma = \bigcap_{\delta<\gamma} W^\delta$. Let $o(W)$ be the least ordinal such that $W^{o(W)} = W^{o(W)+1}$. For $\vec{s} \in W$, let $|s|_W$ be the largest ordinal $\gamma \leq o(W)$ such that $\vec{s} \in W^\gamma$ (this is well-defined; note that $|s|_W = o(W)$ iff $\vec{s} \in W^{o(W)}$).

Lemma 2.2. *For all $\beta < \alpha$, ${V_\beta}^{o(V_\beta)} = \emptyset$.*

Proof. Suppose $\vec{s}$ (in the above notation) is in ${V_\beta}^{o(V_\beta)}$. Let σ be a winning strategy for II in the game $G_{A^c_\beta}$ (where $A^c_\beta = \mathcal{P}_\kappa(\lambda) - A_\beta$). We define an infinite sequence extending $\vec{s}$ as follows. Let z'_{n+1} code a set $S_{z'_{n+1}} \in \mathcal{P}_\kappa(\lambda)$ containing both $S_{\tau_0(z_0,\ldots,z_n)}$ and $S_{\sigma(z_n)}$. Thus, $(z_0,\ldots,z_n,z'_{n+1})$ is good. Since $\vec{s}\in V_\beta^{o(V_\beta)}=V_\beta^{o(V_\beta)+1}$, we may fix z_{n+1}, $y(n+1)$, and $\beta(n+1)$ with $S_{z_{n+1}}\supseteq S_{z'_{n+1}}$ such that $\vec{s}^\frown(z_{n+1},y(n+1),\beta(n+1))\in V_\beta^{o(V_\beta)}$. Continuing, we defining an infinite sequence $(z_0,y(0),\beta(0)),\ldots,(z_i,y(i),\beta(i)),\ldots$ extending $\vec{s}$ which is a path through $V_\beta^{o(V_\beta)}\subseteq V_\beta$. Let $S=\bigcup_i S_{z_i}$. From the definition of σ, $S\notin A_\beta$. By goodness, we also have $S\in A_{\phi_0(y)}\subseteq A_{\beta(0)}\subseteq A_\beta$, a contradiction. ☐

For $\beta<\alpha$ and $\vec{s}\in V_\beta$, let $|\vec{s}|_\beta$ abbreviate $|\vec{s}|_{V_\beta}$.

From lemma 2.2 it follows that for any $\beta<\alpha$ and $\vec{s}\in V_\beta$ (in the previous notation for $\vec{s}$), that there is a z'_{n+1} such that $(z_0,\ldots,z_n,z'_{n+1})$ is good and

$$(*)_\beta\colon\ \text{for any } \vec{s}_2=\vec{s}^\frown(z_{n+1},y(n+1),\beta(n+1))\in V_\beta \text{ such that } S_{z'_{n+1}}\subseteq S_{z_{n+1}},\ |s_2|_\beta<|s|_\beta.$$

Note that (for fixed $\vec{s}$) this depends only on the set $S_{z'_{n+1}}\in\mathcal{P}_\kappa(\lambda)$ coded by z'_{n+1}.

We define now a certain infinite sequence $(z_0,y(0)),(z_1,y(1)),\ldots$, all of whose initial segments satisfy (1) and (2) in the definition of V. Set $t_{-1}=\emptyset$. Assume now that $t_n=((z_0,y(0)),\ldots,(z_n,y(n)))$ has been defined and satisfies (1), (2) in the definition of V. For $\beta<\alpha$ and $\vec{\beta}=(\beta(0),\ldots,\beta(n))$ such that $(y\restriction n+1,\vec{\beta})\in T_\beta$, let $H(\beta,\vec{\beta})$ be the set of all $S\in\mathcal{P}_\kappa(\lambda)$ such that for some (all) z'_{n+1} coding S, $(z_0,\ldots,z_n,z'_{n+1})$ is good and $(z_0,y(0),\beta(0)),\ldots,(z_n,y(n),\beta(n))$, z'_{n+1} satisfy $(*)_\beta$. Thus, $H(\beta,\vec{\beta})\neq\emptyset$. Applying the λ-regularity of κ to this H, there is an $S\in\mathcal{P}_\kappa(\lambda)$ which covers a set in $H(\beta,\vec{\beta})$ for all such $\beta,\vec{\beta}$. Let z_{n+1} be such that $S_{z_{n+1}}=S$. Let $y(n+1)=\tau_1(z_0,\ldots,z_n,z_{n+1})$, and $t_{n+1}=t_n{}^\frown(z_{n+1},y(n+1))$. Clearly, t_{n+1} also satisfies (1), (2) in the definition of V.

Consider now the infinite sequence $(z_0,y(0)),(z_1,y(1)),\ldots$ given by the t_i. By goodness, $y\in P$ and $S\in A_\beta$ where $\beta=\phi_0(y)$ and $S=\bigcup_i S_{z_i}$. Fix an infinite sequence $\vec{\beta}=(\beta(0),\beta(1),\ldots)$ such that $\beta(0)=\beta$ and $(y,\vec{\beta})\in[T]$, and so $(y,\vec{\beta})\in[T_\beta]$ as well. If we let $s_i=((z_0,y(0),\beta(0)),\ldots,(z_i,y(i),\beta(i)))$, then from the definition of the z_n it follows that

$$|s_1|_\beta>|s_2|_\beta>\cdots>|s_i|_\beta>|s_{i+1}|_\beta>\cdots,$$

a contradiction. This completes the proof of theorem 2.1. Λ

3. Review of Generic codes

The proof of theorem 1.1 makes heavy use of the theory of generic codes of uncountable ordinals, developed by Kechris and Woodin [10]. Since this work is as yet unpublished, we review in this section what we need from this theory, in particular, we give the proof from [10] of the existence of a generic coding function.

Let $x\to((x)_0,(x)_1,\ldots)$ be a reasonable recursive bijection between ω^ω and $(\omega^\omega)^\omega$, say the one defined by $(x)_n(m)=x(\langle n,m\rangle)$, where $(n,m)\to\langle n,m\rangle$ is a bijection increasing in each argument.

Theorem 3.1 (Kechris-Woodin; AD). *Let δ be a reliable ordinal. Let $P\subseteq\omega^\omega$, and $\{\phi_m\}_{m\in\omega}$ be a scale on P with each $\phi_m\colon P\to\delta$ and $\phi_0\colon P\overset{onto}{\to}\delta$. Then there is a Lipschitz continuous (i.e. , a strategy for II) function $G\colon\delta^\omega\to\omega^\omega$ such that:*

(1) $\forall\vec{\alpha}=(\alpha(0),\alpha(1),\ldots)\in\delta^\omega\ \forall n\ G_n(\vec{\alpha})\in P$ *and* $\phi_0(G_n(\vec{\alpha}))\leq\alpha(n)$. *Here* $G_n(\vec{\alpha})\doteq(G(\vec{\alpha}))_n$ *is the* n^{th} *real coded by* $G(\vec{\alpha})$.

(2) *If* $\vec{\alpha} = (\alpha(0), \alpha(1), \dots)$ *enumerates an honest set (with respect to* $\{\phi_m\}$*), then* $\forall n\ \phi_0(G_n(\vec{\alpha})) = \alpha(n)$.

Thus, for $\vec{\alpha}$ enumerating an honest set, $G(\vec{\alpha})$ codes a sequence of reals $G_0(\vec{\alpha})$, $G_1(\vec{\alpha}), \dots$, where each $G_n(\vec{\alpha})$ codes (via ϕ_0) the ordinal $\alpha(n)$. Also, G is produced continuously from $\vec{\alpha}$. G is our Kechris-Woodin generic coding function. We will also make explicit use of the component function G_0.

proof of theorem 3.1. We follow the proof of [10]. We show the existence of the component function G_0 (*i.e.* G_0 satisfies (1) and (2) for $n = 0$), from which the existence of a G as required follows by an easy dovetailing argument. Let $T \subseteq (\omega \times \delta)^{<\omega}$ be the tree of the scale $\{\phi_m\}$. Consider the following ordinal game

$$\begin{array}{lcccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \cdots & \\ \mathrm{II} & & \beta_0 & & \beta_1 & & \cdots \\ & & x(0) & & x(1) & & \cdots \end{array}$$

Here α_i, β_i are ordinals $< \delta$, and the $x(i)$ are integers. Let $S = \bigcup_i \{\alpha_i\} \cup \bigcup_i \{\beta_i\}$ be the set of ordinals enumerated by both players, and let $x = (x(0), x(1), \dots)$ be the real played by II. Then II wins the run iff

$$\exists \vec{t} \in S^\omega\ (x, \alpha_0 {}^\frown \vec{t}) \in [T]\ \wedge\ \forall y\ \forall \vec{t} \in S^\omega\ [(y, \alpha_0 {}^\frown \vec{t}) \in [T]) \implies \phi_0(y) \le \phi_0(x)]$$

Assume for the moment the game is determined. Then I cannot have a winning strategy σ, for as soon as σ plays α_0, II picks $x \in P$ with $\phi_0(x) = \alpha_0$, and plays x along with $\beta_i = \phi_i(x)$. This defeats I. A winning strategy τ for II (ignoring the ordinal moves of II) gives the function G_0. It remains to show the game is determined.

For $\vec{s} = (\gamma_0, \gamma_1, \dots) \in \delta^\omega$, and $\pi \in \omega^\omega$, let $\vec{s} \circ \pi \in \delta^\omega$ be the sequence $\vec{s} \circ \pi(i) = \gamma_{\pi(i)}$. If $\vec{s}$, x denotes a run of the game, the payoff for II can be written as

$$\exists \pi \in \omega^\omega\ (x, \alpha_0 {}^\frown \vec{s} \circ \pi) \in [T] \wedge$$
$$\forall y \in \omega^\omega\ \forall \pi \in \omega^\omega\ [(y, \alpha_0 {}^\frown \vec{s} \circ \pi) \in [T]) \implies \phi_0(y) \le \phi_0(x)]$$

The collection of δ-pointsets (*i.e.* subsets of $(\delta^\omega)^m \times (\omega^\omega)^n$ for some m, n) which are both Suslin and co-Suslin is closed under complements, countable unions and intersections, and real quantification (the latter by the 2^{nd} periodicity theorem, which holds for δ-pointsets). The payoff set is then easily Suslin, co-Suslin. A consequence of AD, however is that every ordinal game with Suslin, co-Suslin payoff is determined, and we are done. ☐

Remark 3.1. The determinacy of Suslin, co-Suslin ordinal games is due originally to Moschovakis, see theorem 2.2 of [13]. Another proof, along different lines, is given by Martin and Steel in [11] (the result is implicit in the proof of theorem 2.1 of that paper).

For any $S \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$, the set S^ω carries a topology, the product of the discrete topology on S, with which it is homeomorphic to ω^ω (though the homeomorphism cannot be obtained uniformly in S). We write $\forall^* s \in S^\omega$ to abbreviate "for comeager many s in S^ω." Similarly, if $p \in S^{<\omega}$, we write $\forall^*_p s \in S^\omega$ for "for comeager in the neighborhood N_p many s in S^ω," where N_p is the basic open set $N_p = \{s \in S^\omega : s \text{ extends } p\}$. Recall that from AD every subset of ω^ω has the Baire property, and every wellordered union of meager sets is meager. These facts, of course, also hold for subsets of S^ω. Thus, the generic coding function G_0 provides, uniformly for honest $S \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$ containing α, a notion of category on the codes of α.

4. Below The Projective Ordinals

In this section we prove the following theorem.

Theorem 4.1. *Every projective ordinal $\boldsymbol{\delta}^1_n$ is λ-supercompact for all $\lambda < \aleph_{\aleph_1}$.*

The projective hierarchy analysis shows that $\sup_n \boldsymbol{\delta}^1_n = \aleph_{\epsilon_0} < \aleph_{\aleph_1}$, and thus theorem 4.1 implies theorem 1.1 (recall ϵ_0 is the ordinal defined by $\epsilon_0 = \sup_n \omega(n)$, where $\omega(0) = 1$ and $\omega(n+1) = \omega^{\omega(n)}$). This is our only necessary use of this theory in obtaining theorem 1.1 (we will use one other consequence of that analysis which, however, follows also by the Kechris-Woodin theory of generic codes; we give a proof below). This analysis is outlined in [8], and described in detail in [4] and [7]. The reader can also consult [5] for an overview of the determinacy theory of the projective ordinals (and beyond).

We require the following two technical lemmas, both of which hold in more general forms than we state here. Lemma 4.1 is due to Steel, and is implicit in theorem 3.1 of [14]. It is also proved (in a more general form) in [6], so we omit the proof here. Lemma 4.2 actually holds for any reliable δ (though we don't need this). Recall Θ is the supremum of the lengths of the prewellorderings of $\mathbb{R}$.

Lemma 4.1. *Suppose $\kappa < \Theta$ and $\operatorname{cof}(\kappa) > \boldsymbol{\delta}^1_{2n+1}$. Then there is a $W \subseteq \omega^\omega$ and a norm $\psi \colon W \overset{onto}{\to} \kappa$ which is $\boldsymbol{\Sigma}^1_{2n+2}$ bounded. By this we mean that if $B \subseteq W$ is $\boldsymbol{\Sigma}^1_{2n+2}$, then $\sup\{\psi(x) \colon x \in B\} < \kappa$.*

Lemma 4.2. *Let ν denote the supercompactness measure on $\boldsymbol{\delta}^1_{2n+1}$. Suppose $\kappa < \Theta$, $\operatorname{cof}(\kappa) > \boldsymbol{\delta}^1_{2n+1}$, and $H \colon \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1}) \to \kappa$. Then there is a $\gamma < \kappa$ such that $\forall^*_\nu S$ $H(S) < \gamma$.*

Proof. Let W, ψ be as in lemma 4.1. Let P be a $\boldsymbol{\Pi}^1_{2n+1}$ complete set, and $x \to |x|$ a $\boldsymbol{\Pi}^1_{2n+1}$ norm on P. Consider the following real game G:

$$\begin{array}{lcccccc}
\mathrm{I} & z_0 & & z_2 & & \cdots & \\
\mathrm{II} & & z_1 & & z_3 & & \cdots \\
 & & x(0) & & x(1) & & \cdots
\end{array}$$

Here I and II play reals $z_i \in P$, and II plays integers $x(i)$ as well, building a real x. Let $\alpha_i = |z_i|$, and $S = \{\alpha_i \colon i \in \omega\} \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$. II wins the run of the game iff $x \in W$ and $\psi(x) > H(S)$. Since the payoff depends only on the ordinals α_i coded by the z_i, the Harrington-Kechris theorem [3] shows that the game is determined from AD. Now, I cannot have a winning strategy σ. For if so, then we could easily define a sequence $z_1, z_3, \dots$ of reals in P such that for any $x \in \omega^\omega$, if II plays $z_1, z_3, \dots$ and x against σ, then $S \doteq \{|z_i|\}_{i\in\omega} \subseteq S_2 = \{|z_{2i+1}|\}_{i\in\omega}$ (*i.e.* the ordinals played by I are covered by the ordinals played by II). If II plays $z_1, z_3, \dots$ along with an $x \in W$ such that $\psi(x) > H(S_2)$, then II defeats I, a contradiction. Thus II has a winning strategy in G. The Harrington-Kechris theorem moreover shows that the player with a winning strategy has a $\boldsymbol{\Sigma}^1_{2n+2}$ (in fact, a $\boldsymbol{\Pi}^1_{2n+1}$) winning strategy. That is, the relations $R_m(z_0, z_2, \ldots, z_{2m}, y, a) \Leftrightarrow \tau(z_0, z_2, \ldots, z_{2m}) = (y, a)$ are $\boldsymbol{\Sigma}^1_{2n+2}$. Let τ_1 denote the integer part of τ's response.

Let $B = \{\tau_1 * (z_0, z_2, \ldots)\colon \forall m\ z_{2m} \in P\}$. Then $B \in \boldsymbol{\Sigma}^1_{2n+2}$ and $B \subseteq W$. If $\gamma > \sup\{\psi(x)\colon x \in B\}$, then γ verifies the lemma (note that the supercompactness measure ν is defined by the real part of the game G). □

We require one more result which we state in the next lemma. The result is due to Kechris and Woodin [10], who proved it using generic codes. It also follows from the detailed projective hierarchy analysis (more precisely, the extension of that analysis to $\aleph_{\aleph_1}$). The proof we give below is somewhat different from that of [10] in that we avoid the use of generic elementary embeddings.

Lemma 4.3 (Kechris-Woodin [10]). *For all successor cardinals $\boldsymbol{\delta}^1_{2n+1} < \kappa^+ < \aleph_{\aleph_1}$ we have* $\mathrm{cof}(\kappa^+) > \boldsymbol{\delta}^1_{2n+1}$.

Proof. Since all the cardinals $(\boldsymbol{\delta}^1_{2n+1})^{+n}$ have cofinality $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ (Kunen; *c.f.* theorem 18.2 of [9]), we may assume $\kappa^+ > (\boldsymbol{\delta}^1_{2n+1})^{+\omega}$. Let j_ν be the embedding from the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$. Since easily $j_\nu(\omega_1) \geq \boldsymbol{\delta}^1_{2n+2}$ (in fact, it can be shown $j_\nu(\omega_1) = \boldsymbol{\delta}^1_{2n+2}$), j_ν moves all cardinals between ω_1 and $\boldsymbol{\delta}^1_{2n+1}$. It follows that if $\delta \in \mathrm{ON}$ with $\omega_1 \leq \mathrm{cof}(\delta) \leq \boldsymbol{\delta}^1_{2n+1}$, then $j_\nu(\delta) > \delta$. Thus, it suffices to prove by induction on the cardinals λ with $(\boldsymbol{\delta}^1_{2n+1})^{+\omega} \leq \lambda < \aleph_{\aleph_1}$ that $j_\nu(\lambda) = \lambda$.

First consider $\lambda = (\boldsymbol{\delta}^1_{2n+1})^{+\omega}$. We follow the argument of [10]. Let ν' denote the supercompactness measure on $\mathcal{P}_{\omega_1}((\boldsymbol{\delta}^1_{2n+1})^-)$. We first note that Becker has shown that $j_{\nu'}(\boldsymbol{\delta}^1_{2n+1}) < \boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ [equivalence classes $[F]_{\nu'}$ of functions $F\colon \mathcal{P}_{\omega_1}((\boldsymbol{\delta}^1_{2n+1})^-) \to \boldsymbol{\delta}^1_{2n+1}$ can be coded via $\boldsymbol{\Delta}^1_{2n+1}$ strategies for games similar to lemma 4.2, using a $\boldsymbol{\Delta}^1_{2n+1}$ prewellordering of length $(\boldsymbol{\delta}^1_{2n+1})^-$. The prewellordering on the set of codes of functions corresponding to the ultrapower relation can be computed to be $\boldsymbol{\Sigma}^1_{2n+2}$]. Let μ_ω be the ω-cofinal normal measure on $\boldsymbol{\delta}^1_{2n+1}$. For any ordinal θ, $j_\nu(\theta) \leq j_{\mu_\omega}(j_{\nu'}(\theta))$ [the measure $\hat{\nu}$ on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$ defined by $\hat{\nu}(A) = 1$ iff $\forall^*_{\mu_\omega} \alpha\ \forall^* S \in \mathcal{P}_{\omega_1}(\alpha)\ S \in A$ defines a fine, normal measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$. By Woodin's theorem [16] on the uniqueness of the supercompactness measure, $\hat{\nu} = \nu$]. So $j_\nu(\boldsymbol{\delta}^1_{2n+1}) < j_{\mu_\omega}((\boldsymbol{\delta}^1_{2n+1})^+) = (\boldsymbol{\delta}^1_{2n+1})^{++}$. Since $j_{\mu_\omega}((\boldsymbol{\delta}^1_{2n+1})^n) = (\boldsymbol{\delta}^1_{2n+1})^{n+1}$, a straightforward induction using lemma 4.2 and the fact that $\mathrm{cof}((\boldsymbol{\delta}^1_{2n+1})^{+n}) = \boldsymbol{\delta}^1_{2n+2}$ shows $j_\nu((\boldsymbol{\delta}^1_{2n+1})^{+n}) \leq (\boldsymbol{\delta}^1_{2n+1})^{+n+1}$ for all $n \geq 1$ (in fact, equality must hold here as $j_\nu \geq j_{\mu_\omega}$). Thus, $j_\nu(\lambda) = \lambda$.

If λ is a limit cardinal, then $\mathrm{cof}(\lambda) = \omega$, and the result is immediate by induction.

Finally, suppose $\lambda = \kappa^+$. By induction, $j_\nu(\kappa) = \kappa$. We show directly in this case that $\mathrm{cof}(\kappa^+) > \boldsymbol{\delta}^1_{2n+1}$, using an argument similar to that of theorem 3.2 of [6]. Suppose, towards a contradiction, that $f\colon \boldsymbol{\delta}^1_{2n+1} \to \kappa^+$ is cofinal. We obtain a contradiction by producing a function H with domain $\boldsymbol{\delta}^1_{2n+1}$ such that for all $\alpha < \boldsymbol{\delta}^1_{2n+1}$, $H(\alpha)$ is a wellordering of κ of length $\geq f(\alpha)$. Let $P \in \boldsymbol{\Pi}^1_3$, $\{\phi_m\}$, G, G_0 be as in theorem 3.1. By the coding lemma, there is a function F with domain P such that for all $x \in P$, $F(x)$ codes a wellordering of κ of length $f(|x|)$ (the exact nature of this coding is irrelevant, say by the coding lemma with respect to a suitable pointclass). For $\vec{s} \in (\boldsymbol{\delta}^1_{2n+1})^\omega$, let $\prec_{\vec{s}}$ be the wellordering $F(G_0(\vec{s}))$. Fix for the moment $\alpha < \boldsymbol{\delta}^1_{2n+1}$. For honest $S \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$ containing α, we define a tree $V_{\alpha,S} \subseteq ((\boldsymbol{\delta}^1_{2n+1})^{<\omega} \times \kappa)^{<\omega}$ by: $((p_0, \ldots, p_n), (\beta_0, \ldots, \beta_n)) \in V_{\alpha,S}$ iff

(1) $\forall i < n\ p_{i+1}$ extends p_i.
(2) $\forall i < n\ \forall^*_{p_{i+1}} \vec{s} \in S^\omega\ \beta_{i+1} \prec_{\alpha^\frown \vec{s}} \beta_i$
(3) $\forall i \leq n\ \exists \gamma_i\ (\forall^*_{p_i} \vec{s} \in S^\omega\ |\beta_i|_{\prec_{\alpha^\frown \vec{s}}} = \gamma_i)$.

(recall $\forall^*_p \vec{s} \in S^\omega$ means "for comeager in the neighborhood N_p many $\vec{s} \in S^\omega$"). Using (3), it is easy to see that $V_{\alpha,S}$ is wellfounded. We show that $|V_{\alpha,S}| \geq f(\alpha)$. Suppose $|V_{\alpha,S}| \doteq \theta_0 < f(\alpha)$. Since $\forall^* \vec{s} \in S^\omega\ |\prec_{\alpha^\frown \vec{s}}| = f(\alpha) > \theta_0$, by additivity of category and Baire measurability we get a $p_0 \in S^{<\omega}$, a $\beta_0 < \kappa$, and a γ_0 such that $\forall^*_{p_0} \vec{s} \in S^\omega\ |\beta_0|_{\prec_{\alpha^\frown \vec{s}}} = \gamma_0 \geq \theta_0$. Thus, $(p_0, \beta_0) \in V_{\alpha,S}$, and $\forall^*_{p_0} \vec{s} \in S^\omega\ |\beta_0|_{\prec_{\alpha^\frown \vec{s}}} > \theta_1 \doteq |(p_0,\beta_0)|_{V_{\alpha,S}}$. Continuing, we produce a branch through $V_{\alpha,S}$, a contradiction.

The function $S \to V_{\alpha,S}$ represents, in the ultrapower by ν, a wellordering (actually a wellfounded tree) of $j_\nu(\kappa) = \kappa$ of length at least $f(\alpha)$. □

We turn now to the proof of theorem 4.1. For the remainder of this section, λ will denote a fixed cardinal less than $\aleph_{\aleph_1}$.

As we remarked in the introduction, all of the $\boldsymbol{\delta}^1_n$ are regular, and $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$. Also, each $\boldsymbol{\delta}^1_{2n+1}$ is a Suslin cardinal, and $\aleph_{\aleph_1}$ is less than the supremum of the Suslin cardinals. By theorem 2.1, to show that the $\boldsymbol{\delta}^1_n$ are λ-supercompact, it suffices to show each $\boldsymbol{\delta}^1_n$ is λ-regular. From fact (c) mentioned at the beginning of §2, it suffices to show that $\boldsymbol{\delta}^1_{2n+2}$ is λ-regular. In verifying definition 2.1, we may clearly assume then that $\alpha = \boldsymbol{\delta}^1_{2n+1}$.

Thus, we are in the following setting. We are given $f\colon \boldsymbol{\delta}^1_{2n+1} \to \mathcal{P}(\mathcal{P}_{\boldsymbol{\delta}^1_{2n+2}}(\lambda)) - \{\emptyset\}$, and we must find a set $A \subseteq \lambda$ with $|A| \leq \boldsymbol{\delta}^1_{2n+1}$ which covers a set in each $f(\beta)$, $\beta < \boldsymbol{\delta}^1_{2n+1}$. That is, $\forall \beta < \boldsymbol{\delta}^1_{2n+1}\ \exists B \in f(\beta)\ B \subseteq A$. This function f is fixed for the rest of this section.

Fix a $\boldsymbol{\Pi}^1_{2n+1}$-complete set P and a $\boldsymbol{\Pi}^1_{2n+1}$ scale $\{\phi_m\}$ on P, with ϕ_0 onto $\boldsymbol{\delta}^1_{2n+1}$. For $x \in P$ we write $|x|$ in place of $\phi_0(x)$. We fix for the rest of the proof the generic coding functions G, G_0 given by theorem 3.1.

Let ν denote the supercompactness measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$. We write $\forall^*_\nu S$ to abbreviate "for ν almost all $S \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$." Recall we write $\forall^*_p s \in S^\omega$ for "for comeager in the neighborhood N_p many s in S^ω," where N_p is the basic open set $N_p = \{s \in S^\omega\colon s \text{ extends } p\}$.

By the coding lemma, there is a $\boldsymbol{\Sigma}^1_{2n+2}$ relation $R \subseteq \omega^\omega \times \omega^\omega$ with $\mathrm{dom}(R) = P$ and $\forall x, y\ [R(x,y) \to S_y \in f(|x|)]$. Recall $S_y \in \mathcal{P}_{\boldsymbol{\delta}^1_{2n+2}}(\lambda)$ is the set coded by y. Let $F'\colon \omega^\omega \to \omega^\omega$ uniformize R, that is, $\forall x \in P\ S_{F'(x)} \in f(|x|)$. For $x \in P$, let $F(x) = S_{F'(x)}$. We may summarize the properties of F, G by: $\forall \alpha < \boldsymbol{\delta}^1_{2n+1}\ \forall^*_\nu S\ \forall^* s \in S^\omega\ F(G_0(\alpha^\frown s)) \in f(\alpha)$.

For $\alpha < \boldsymbol{\delta}^1_{2n+1}$ and $\vec{s} \in (\boldsymbol{\delta}^1_{2n+1})^\omega$, let $A_{\alpha^\frown \vec{s}}$ denote $F(G_0(\alpha^\frown \vec{s}))$. Thus, if $\vec{s}$ enumerates an honest set containing α, then $A_{\alpha^\frown \vec{s}} \in f(\alpha)$.

Let $\mathcal{S}$ denote the set of successor cardinals κ with $\boldsymbol{\delta}^1_{2n+1} < \kappa \leq \lambda$. Thus, $\mathcal{S}$ is countable. Suppose for the moment $\kappa \in \mathcal{S}$. Then if $\alpha < \boldsymbol{\delta}^1_{2n+1}$ and an honest $S \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})$ containing α are fixed, additivity of category and the fact that $\mathrm{cof}(\kappa) > \boldsymbol{\delta}^1_{2n+1}$ imply that $\exists \gamma < \kappa\ \forall^* \vec{s} \in S^\omega\ A^\kappa_{\alpha^\frown \vec{s}} \subseteq \gamma$, where $A^\kappa_{\alpha^\frown \vec{s}} = A_{\alpha^\frown \vec{s}} \cap \kappa$. Since $\mathrm{cof}(\kappa) > \boldsymbol{\delta}^1_{2n+1}$ by lemma 4.3, it then follows from lemma 4.2 that $\forall \alpha < \boldsymbol{\delta}^1_{2n+1}\ \exists \gamma < \kappa\ \forall^*_\nu S\ \forall^* \vec{s} \in S^\omega\ A^\kappa_{\alpha^\frown \vec{s}} \subseteq \gamma$. Finally, using again $\mathrm{cof}(\kappa) > \boldsymbol{\delta}^1_{2n+1}$ it follows that $\exists \gamma < \kappa\ \forall \alpha < \boldsymbol{\delta}^1_{2n+1}\ \forall^*_\nu S\ \forall^* \vec{s} \in S^\omega\ A^\kappa_{\alpha^\frown \vec{s}} \subseteq \gamma$.

For each $\kappa \in \mathcal{S}$ we define a sequence of ordinals $\kappa^0 \leq \kappa^1 \leq \dots$, where $\kappa^- < \kappa^i < \kappa$ for all i. By the previous paragraph, let κ^0 be the least ordinal $> \kappa^-$ such that $\forall \alpha\ \forall^*_\nu S\ \forall^* \vec{s}\ A^\kappa_{\alpha^\frown \vec{s}} \subseteq \kappa^0$. Thus, $\kappa^0 < \kappa$. By countable choice, fix for each $\kappa \in \mathcal{S}$ a bijection $\pi^0_\kappa \colon \kappa^0 \to \kappa^-$. By countable additivity we have $\forall \alpha\ \forall^*_\nu S\ \forall^* \vec{s}\ \forall \kappa \in \mathcal{S}$ $A^\kappa_{\alpha^\frown \vec{s}} \subseteq \kappa^0$. If $E \subseteq \lambda$, define $\Pi_0(E) = \bigcup_{\kappa \in \mathcal{S}} \pi^0_\kappa(E \cap \kappa^0)$.

In general, suppose that for each $\kappa \in \mathcal{S}$ the ordinals $\kappa^0, \dots, \kappa^m$ and bijections $\pi^i_\kappa \colon \kappa^i \to \kappa^-$ have been defined. For $E \subseteq \lambda$, let $\Pi_i(E) = \bigcup_{\kappa \in \mathcal{S}} \pi^i_\kappa(E \cap \kappa^i)$. Assume inductively that

$$\forall \alpha\ \forall^*_\nu S\ \forall^* \vec{s}\ \forall i \leq m\ \forall \kappa \in \mathcal{S}\ (\Pi_{i-1} \circ \dots \circ \Pi_0(A_{\alpha^\frown \vec{s}})) \cap \kappa \subseteq \kappa^i$$

(where we regard Π_{-1} as the identity). We apply the argument of the previous paragraph to the sets $\Pi_m \circ \dots \circ \Pi_0(A_{\alpha^\frown \vec{s}})$. This produces ordinals $\kappa^{m+1} < \kappa$ such that $\forall \alpha\ \forall^*_\nu S\ \forall^* \vec{s}\ \forall \kappa \in \mathcal{S}\ \Pi_m \circ \dots \circ \Pi_0(E) \cap \kappa \subseteq \kappa^{m+1}$. We may assume $\kappa^{m+1} > \kappa^m$. By countable choice we may fix bijections $\pi^{m+1}_\kappa \colon \kappa^{m+1} \to \kappa^-$.

Thus, we have defined κ^m and π^m_κ for all $\kappa \in \mathcal{S}$ and $m \in \omega$. Let $\mathcal{T}$ be the tree of all finite decreasing sequences $\kappa_0 > \kappa_1 > \dots > \kappa_j$ of successor cardinals in $\mathcal{S}$ with $\kappa_j = \boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$. For $\vec{\rho} = (\kappa_0, \dots, \kappa_j) \in \mathcal{T}$, define $A_{\vec{\rho}} = {\pi^0_{\kappa_0}}^{-1}(\dots({\pi^j_{\kappa_j}}^{-1}(\boldsymbol{\delta}^1_{2n+1})))$. Finally, set $A = \bigcup_{\vec{\rho} \in \mathcal{T}} A_{\vec{\rho}}$. Clearly $|A| = \boldsymbol{\delta}^1_{2n+1}$. We show that A covers f as desired. Fix $\alpha < \boldsymbol{\delta}^1_{2n+1}$. For each $m \in \omega$, let M_m be a ν measure one set such that for all $S \in M_m$

$$\forall^* \vec{s} \in S^\omega\ \forall \kappa \in \mathcal{S}\ (\Pi_{m-1} \circ \dots \circ \Pi_0(A_{\alpha^\frown \vec{s}})) \cap \kappa \subseteq \kappa^m.$$

Let $M = \bigcap_m M_m$, so $\nu(M) = 1$. Fix an honest set $S \in M$ containing α. It suffices to show that $\forall^* \vec{s} \in S^\omega\ A_{\alpha^\frown \vec{s}} \subseteq A$. Fix then a $\vec{s} \in S^\omega$ so that

$$\forall m \in \omega\ \forall \kappa \in \mathcal{S}\ (\Pi_{m-1} \circ \dots \circ \Pi_0(A_{\alpha^\frown \vec{s}})) \cap \kappa \subseteq \kappa^m. \tag{1}$$

We show that $A_{\alpha^\frown \vec{s}} \subseteq A$. Let $\beta_0 \in A_{\alpha^\frown \vec{s}}$, and let $\kappa_0 \in \mathcal{S}$ with $\beta_0 < \kappa_0$. By (1), $\beta_0 < \kappa^0_0$. Let $\beta_1 = \pi^0_{\kappa_0}(\beta_0)$. Let $\kappa_1 \in \mathcal{S}$, $\kappa_1 < \kappa_0$ be such that $\beta_1 < \kappa_1$. By (1) again, $\beta_1 < \kappa^1_1$. Continuing, we produce sequences $\beta_0, \beta_1, \dots, \beta_j$ and $\kappa_0 > \kappa_1 > \dots > \kappa_j = \boldsymbol{\delta}^1_{2n+2}$. Thus, $\beta_j < \kappa^j_j$, and $\beta_{j+1} = \pi^j_{\kappa_j}(\beta_j) < \boldsymbol{\delta}^1_{2n+1}$. Hence, $\beta_0 = {\pi^0_{\kappa_0}}^{-1} \circ \dots \circ {\pi^j_{\kappa_j}}^{-1}(\beta_{j+1})$, and so $\beta_0 \in A_{\vec{\rho}}$ where $\vec{\rho} = (\kappa_0, \dots, \kappa_j)$. This completes the proof of theorem 4.1. Λ

In conclusion, we mention that the methods of this paper can be used to extend theorem 4.1 to large values of λ, but doing so seems to require use of the detailed (extension of the) projective hierarchy analysis. Using this analysis, and the methods of this paper, we can extend theorem 4.1 to $\lambda < \theta \doteq$ the least cardinal of uncountable cofinality closed under the $\aleph$ function (*i.e.* the function $\alpha \to \aleph_\alpha$). As we mentioned earlier, the more recent results of [6] provide a still further extension without recourse to this theory.

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