THE WEAK SQUARE PROPERTY

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Abstract. We formulate and prove a combinatorial property assuming $AD + V = L(\mathbb{R})$. As a consequence, we show that every regular κ which is either a Suslin cardinal or the successor of a Suslin cardinal is $\boldsymbol{\delta}^2_1$-supercompact. In particular, all the projective ordinals $\boldsymbol{\delta}^1_n$ are $\boldsymbol{\delta}^2_1$-supercompact.

1. Introduction

We work throughout in the theory $AD + V = L(\mathbb{R})$ (recall Kechris has shown that DC follows), although some of our results are true more generally (see remark 2.1). Our main purpose is to consider a combinatorial property which we call the weak square property. Specifically, for cardinals $\kappa < \lambda$, let $\boxminus_{\kappa,\lambda}$ denote the assertion: for any $f : \kappa \to \lambda$ such that $\forall \alpha < \kappa \ \operatorname{cof}(f(\alpha)) \leq \kappa$, there is an $A \subseteq \lambda$, $|A| \leq \kappa$, such that $\forall \alpha < \kappa \ (A \cap f(\alpha)$ is cofinal in $f(\alpha))$.

The main result of this paper is that, for κ a Suslin cardinal, $\boxminus_{\kappa,\lambda}$ holds for all $\lambda < \Theta$. Recall Θ denotes the supremum of the lengths of the prewellorderings of the reals. Also, κ is a Suslin cardinal if there is a set $A \subseteq \omega^\omega$ which is κ-Suslin but not κ'-Suslin for any $\kappa' < \kappa$ (recall A is κ-Suslin if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $A = p[T]$, that is, $x \in A$ iff $\exists f \in \kappa^\omega \ \forall n \ (x \restriction n, f \restriction n) \in T)$.

This has a number of consequences. In [3] it is shown that all of the projective ordinals $\boldsymbol{\delta}^1_n$ are $\aleph_{\aleph_1}$-supercompact (the projective hierarchy analysis shows that $\sup_n \boldsymbol{\delta}^1_n = \aleph_{\epsilon(0)} < \aleph_{\aleph_1}$, where $\epsilon(0) = \sup_n \omega(n)$, and $\omega(n)$ is defined by: $\omega(0) = 1$, $\omega(n+1) = \omega^{\omega(n)}$). As a consequence of the weak square property we show in this paper that for every regular cardinal $\kappa \leq \boldsymbol{\delta}^2_1$ which is either a Suslin cardinal or the successor of a Suslin cardinal, κ is $\boldsymbol{\delta}^2_1$-supercompact. Recall (*c.f.* [8]) $\boldsymbol{\delta}^2_1$ is the largest Suslin cardinal in $L(\mathbb{R})$. In particular, the projective ordinals $\boldsymbol{\delta}^1_n$ are all $\boldsymbol{\delta}^2_1$-supercompact. This was previously known for $\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 = \omega_2$. The former is a theorem of Solovay (who assumed $AD^{\mathbb{R}}$; the reduction to AD followed from an ordinal determinacy result of Harrington and Kechris [5], combined with the Martin-Steel theorem [8] that $\boldsymbol{\Sigma}^2_1$ has the scale property in $L(\mathbb{R})$), and the latter a theorem of Becker [1] (see also [3] where another proof is given). The arguments of this paper do not depend on these previous results.

The weak square property also seems to be important in extending the detailed $L(\mathbb{R})$ analysis to higher levels of the $L(\mathbb{R})$ hierarchy, and in fact this was another motivation for formulating it (this is discussed in a little more detail at the end of [6]).

Our arguments here are reasonably self-contained modulo basic determinacy theory for which we refer the reader to [9]. In particular, we do not appeal to the detailed projective hierarchy analysis. We do make heavy use of the Kechris-Woodin theory of generic codes for uncountable ordinals [7], but even here we need only the most basic fact from that analysis, which we review below. In getting the most general form of some of our results we need to appeal to some facts from the scale analysis in $L(\mathbb{R})$ [12] (though this is not necessary for our main result, theorem 2.1). The reader wishing to avoid this can restrict their attention to Suslin cardinals κ within the projective hierarchy. For example, the proof that the $\boldsymbol{\delta}^1_n$ are all $\boldsymbol{\delta}^2_1$-supercompact does does need to appeal to this analysis. We also present some applications of these results to the theory of the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\kappa)$.

We fix a reasonable recursive bijection $x \to (x_0, x_1, \dots)$ from ω^ω to $(\omega^\omega)^\omega$, say the one defined by $x_m(n) = x(\langle m, n\rangle)$, where $(m, n) \to \langle m, n\rangle$ is a recursive bijection from $(\omega)^2$ to ω increasing in both arguments. We also let $(x, y) \to \langle x, y\rangle$ denote a recursive bijection from $\omega^\omega \times \omega^\omega \to \omega^\omega$, or from $\omega \times \omega^\omega$ to ω^ω, or from $\omega \times \omega$ to ω, depending on the context. Let $\overline{0} \in \omega^\omega$ denote the constant 0 real (by "real" we always mean an element of ω^ω).

For the rest of this section κ will denote a Suslin cardinal. It follows from the scale analysis in $L(\mathbb{R})$ (*c.f.* theorem 4.3 of [12]) that there is a κ-Suslin set $P \subseteq \omega^\omega$ and a κ-Suslin scale $\{\psi_n\}_{n\in\omega} : P \to \kappa$, with ψ_0 onto κ. We fix such P, $\{\psi_n\}_{n\in\omega}$. For $x \in P$, we let $|x|$ abbreviate $\psi_0(x)$. Recall a set $S \in \mathcal{P}_{\omega_1}(\kappa)$ is ***honest*** if $\forall \alpha \in S\ \exists x \in P\ (\psi_0(x) = \alpha \wedge \forall n\ \psi_n(x) \in S)$. The main lemma of [7] (a proof is also given in [3]) says that there is a continuous function $G : \kappa^\omega \to \omega^\omega$ (in fact, Lipschitz continuous, i.e., a strategy for II) such that for all $s \in \kappa^\omega$ $G(s)_i \in P$ for all i, and if s enumerates an honest set $S \in \mathcal{P}_{\omega_1}(\kappa)$, then $\{|G(s)_0|, |G(s)_1|, \dots\} = s$. We will also make explicit use of the component function G_0. Henceforth, when we discuss a Suslin cardinal κ, we assume these generic coding functions G, G_0 are fixed.

For each $S \in \mathcal{P}_{\omega_1}(\kappa)$, there is a natural topology on S^ω (the product of the discrete topology on S) with which it is homeomorphic to ω^ω (though this homeomorphism cannot be obtained uniformly in S). When considering topological notions on S^ω such as category, we are always referring to this topology. Recall that AD implies that every subset of ω^ω (or S^ω) has the Baire property, and every well-ordered union of meager sets is meager.

From AD, there is a fine normal measure ν on $\mathcal{P}_{\omega_1}(\kappa)$, and in fact by [13] it is unique. We let j_ν denote the embedding from the ultrapower by ν. We let $\forall^* S$ abbreviate "for ν almost all S". We let $\forall^* S\ \forall^* s_1, \dots, s_n \in S^\omega$ abbreviate "for ν almost all S, for comeager many $s_1 \in S^\omega$, ..., for comeager many $s_n \in S^\omega$". We usually just write $\forall^* S\ \forall^* s_1, \dots, s_n$. If $p \in S^{<\omega}$, we write $\forall^*_p s \in S^\omega$ to abbreviate "for comeager in the neighborhood N_p many s in S^ω." Here N_p denotes the basic open set of all $s \in S^\omega$ extending p.

We introduce a notational convention. If $\alpha \in \mathrm{ON}$, and $Q \subseteq \mathrm{ON}$, we write $\forall^* S\ Q(\alpha(S))$ to abbreviate: if we fix a function $S \to \alpha(S)$ representing α with respect to ν, then $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ Q(\alpha(S))$.

We code strategies (Lipschitz continuous functions) from κ^ω to $\kappa^\omega \times \omega^\omega$ by coding the graph of the strategy, as a subset of $\kappa^{<\omega} \times \kappa \times \omega$, using the coding lemma relative to the pointclass of κ-Suslin sets and the prewellordering induced by ψ_0. Note that if z codes such a strategy, then the relation $z(\alpha_0, \dots, \alpha_n) = (\beta_n, a_n)$ is κ-Suslin in the codes (identifying z with the strategy it codes). For z coding a strategy from κ^ω to $\kappa^\omega \times \omega^\omega$, and $s \in \kappa^\omega$, we write $z(s)$ to denote the real obtained by the integer part of II's response following z against I's play of s.

We deal frequently in this paper with strategies z such that for honest $S \in \mathcal{P}_{\omega_1}(\kappa)$ closed under z, and $s \in S^\omega$ enumerating S, $z(s)$ codes a comeager set

$A_{z(s)} \subseteq S^\omega$ and a continuous function $f_{z(s)} : A_{z(s)} \to \omega^\omega$. By this we mean z codes a strategy from κ^ω to $\kappa^\omega \times \omega^\omega$ such that for all honest S closed under z, and $s = (\alpha_0, \alpha_2, \alpha_4, \dots)$ enumerating S, if $(\alpha_1, \alpha_3, \dots)$ is the ordinal part of z's play against s, and $x \in \omega^\omega$ is the integer part, then x_0 codes a comeager set $A \subseteq S^\omega$, and x_1 codes a continuous function $f : A \to \omega^\omega$. Also, x_0 codes A by having each $(x_0)_n$ code a dense open $D_n \subseteq S^\omega$ such that $A = \bigcap\limits_n D_n$. To say y codes the dense open set $D \subseteq S^\omega$ means each $y(k)$ codes a sequence $u_k \in \omega^{<\omega}$, and $D = \bigcup\limits_k N_{u_k^*}$, where if $u_k = (a_0, \dots, a_l)$, then $N_{u_k^*}$ is the basic open set in S^ω determined by the sequence $u_k^* = (\alpha_{a_0}, \dots, \alpha_{a_l})$. Likewise, x_1 codes f by coding a sequence of tuples of integers $(a_0, \dots, a_l, b_0, \dots, b_m)$, and for $s \in A \cap N_{u^*}$, $u = (a_0, \dots, a_l)$, we have $f(s)$ extends $(b_0, \dots, b_m)$. It is easy to check that for any fixed $S \in \mathcal{P}_{\omega_1}(\kappa)$, and any $s \in S^\omega$ enumerating S, the set of $x \in \omega^\omega$ coding a comeager set A_x and a continuous function $f_x \colon A_x \to \omega^\omega$ relative to s is a $\boldsymbol{\Pi}^0_2$ set.

The next lemma embodies a general ordinal determinacy result which we will use frequently in this paper. Similar results appear in [7].

Lemma 1.1. *Let $\delta < \Theta$, $A \subseteq \omega^\omega$, and $G \colon \delta^\omega \times (\omega^\omega)^n \to \omega^\omega$ be continuous. Let G_A be the ordinal game where I, II alternate playing ordinals $\alpha(0), \alpha(1), \dots < \delta$ jointly building the sequence $\vec{\alpha} = (\alpha(0), \alpha(1), \dots)$, and II wins the run of the game iff $\exists y_1\ \forall y_2 \cdots \exists(\forall) y_n\ A(G(\vec{\alpha}, y_1, \dots, y_n))$. Then G_A is determined.*

Proof. We employ a reflection argument due to Solovay. Suppose the theorem fails. Then (in $L(\mathbb{R})$) there is an $M \subseteq \omega^\omega$ and an $E \subseteq M^2$ such that (M, E) is well-founded, models $ZF_N + AD + V = L(\mathbb{R})$ for sufficiently large N, $\mathbb{R} \subseteq \pi(M)$, where π denotes the transitive collapse of (M, E), and $(M, E) \models$"$\exists \delta, A, G$ witnessing the theorem fails." Since this is a $\boldsymbol{\Sigma}^2_1$ (in fact projective) statement about M, E, by the reflection principle we can find such a pair (M, E) with $M, E \in \boldsymbol{\Delta}^2_1$. We fix such M, E, and the witnesses δ, A, G. Let $\pi \colon (M, E) \cong L_\gamma(\mathbb{R})$ be the transitive collapse, so $\gamma < \boldsymbol{\delta}^2_1$. Let δ', A', G' denote the images of δ, A, G under π. In particular, $A' \in \boldsymbol{\Delta}^2_1$, and thus is Suslin, co-Suslin by [8]. Thus, $\{(\vec{\alpha}, y_1, \dots, y_n) \in (\delta')^\omega \times (\omega^\omega)^n \colon A'(G'(\vec{\alpha}, y_1, \dots, y_n))\}$ is also Suslin, co-Suslin. From the second periodicity theorem, which implies the propagation of scales under real quantification, it follows that $B = \{\vec{\alpha} \in (\delta')^\omega \colon \exists y_1 \dots \exists(\forall) y_n\ A'(G'(\vec{\alpha}, y_1, \dots, y_n))\}$ is also Suslin, co-Suslin. Since Suslin, co-Suslin games are determined (*c.f.* theorem 2.2 of [10] for the original proof, or theorem 2.1 of [8] for another), it follows that B is determined in $L(\mathbb{R})$. By the coding lemma, any strategy for B lies in $L_\gamma(\mathbb{R})$, since it is essentially a subset of $\delta' < \Theta^{L_\gamma(\mathbb{R})}$. This is easily a contradiction. $\square$

All of the ordinal games used in this paper can be viewed as having a payoff of the form $A(G(\vec{\alpha}))$, where $A \subseteq \omega^\omega$, and G is continuous, a special case of lemma 1.1.

The following theorem of Steel (*c.f.* theorem 3.1 of [11] and also theorem 7.4 of [2]) will be used throughout. For the convenience of the reader, we sketch the proof.

Theorem 1.1 (Steel; AD)**.** *Suppose $\alpha < \Theta$ is a limit ordinal. Then there is a set $C \subseteq \omega^\omega$ and a norm $\phi \colon C \overset{onto}{\to} \alpha$ which is κ'-Suslin bounded for all Suslin cardinals $\kappa' < \mathrm{cof}(\alpha)$. That is, if $A \subseteq C$ and A is κ'-Suslin for some $\kappa' < \mathrm{cof}(\alpha)$, then $\sup \{\phi(z) : z \in A\} < \alpha$.*

Proof. We may assume $\text{cof}(\alpha) > \omega$, as otherwise the result is trivial. We may also assume α is regular, since a prewellordering of length $\text{cof}(\alpha)$ which is κ'-Suslin bounded for all $\kappa' < \text{cof}(\alpha)$ easily produces one of length α. For example, let $\delta = \text{cof}(\alpha)$, $h\colon \delta \to \alpha$ be cofinal and increasing, and $\psi\colon A \to \delta$ a norm which is κ'-Suslin bounded for all $\kappa' < \text{cof}(\alpha)$. Let $\phi : B \overset{onto}{\to} \alpha$ be a norm. Define

$$x \in C \leftrightarrow [x_0 \in B \ \wedge x_1 \in A \ \wedge \ \phi(x_0) < h(\psi(x_1))].$$

For $x \in C$, let $\eta(x) = \phi(x_0)$. Then $\eta\colon C \to \alpha$ is κ'-Suslin bounded for all $\kappa' < \text{cof}(\alpha)$. Likewise, it suffices to produce such a prewellordering on some ordinal ρ of cofinality α. [Suppose $h\colon \alpha \to \rho$ is cofinal. If $\phi\colon A \to \rho$ is κ'-Suslin bounded for all $\kappa' < \text{cof}(\alpha)$, then $\psi\colon A \to \alpha$ is also, where $\psi(x) =$ the least $\beta < \alpha$ such that $h(\beta) > \phi(x)$.]

Let $\boldsymbol{\Delta}$ be a pointclass closed under complements and real quantification, and with $o(\boldsymbol{\Delta}) = \rho$, where $\text{cof}(\rho) = \alpha$. Let $\boldsymbol{\Gamma}$ be the non-selfdual pointclass closed under $\forall^{\omega^\omega}$ such that $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \tilde{\boldsymbol{\Gamma}}$ (see [11], here $\tilde{\boldsymbol{\Gamma}}$ denotes the complementary pointclass). By theorem 2.1 of [11], $\tilde{\boldsymbol{\Gamma}}$ is closed under intersections with κ'-Suslin sets for all $\kappa' < \text{cof}(\rho) = \alpha$. Let $h\colon \alpha \to \rho$ be cofinal. Let A be a $\boldsymbol{\Gamma}$-complete set. Define B by: $x \in B$ iff x_0, x_1 code continuous function f_{x_0}, f_{x_1} and $f_{x_0}^{-1}(A) = f_{x_1}^{-1}(\omega^\omega - A)$. For $x \in B$, let $\phi(x)$ be the Wadge degree of the $\boldsymbol{\Delta}$ set $f_{x_0}^{-1}(A)$. Clearly $\phi\colon B \to \rho$. Suppose $S \subseteq B$ is κ'-Suslin for some $\kappa' < \alpha$, and S is unbounded in ϕ. Define

$$C(x,y) \leftrightarrow (x \in S \wedge y \in f_{x_0}^{-1}(A)) \leftrightarrow (x \in S \wedge y \in f_{x_1}^{-1}(\omega^\omega - A)).$$

Then $C \in \boldsymbol{\Delta}$. This is a contradiction, however, since we can clearly reduce any set in $\boldsymbol{\Delta}$ to C (by fixing $x \in S$ so that $f_{x_0}^{-1}(A)$ has Wadge degree above the given set). $\square$

We mention now a coding of the ordinals that we will be using throughout the paper. Let $\lambda < \Theta$, and let $\boldsymbol{\Gamma}$ be a pointclass of sufficiently high Wadge degree above λ. We say then that $u \in \omega^\omega$ codes $\alpha < \lambda$, where $\text{cof}(\alpha) > \omega$, if u codes, via a universal $\boldsymbol{\Gamma}$ set, a prewellordering (C_u, ϕ_u) of length α which is $\boldsymbol{\Sigma}_1^1$-bounded. If $\text{cof}(\alpha) = \omega$, then we require each u_i code $\alpha_i < \alpha$, via some fixed prewellordering of length λ, such that $\sup_i \alpha_i = \alpha$. Occasionally, when coding α with $\text{cof}(\alpha) > \kappa$ (κ a Suslin cardinal) we will require the prewellordering (C_u, ϕ_u) to be κ-Suslin bounded. In all cases, the background pointclass $\boldsymbol{\Gamma}$ is implicitly being used, although generally irrelevant to the argument.

In the proof of the main result given in the next section, the main idea involves a pressing down argument with respect to functions on the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$ and category quantifiers, as well as a technique for handling such functions. To illustrate this technique in a simpler setting, we consider the following theorem (which will also be used later).

Theorem 1.2. *Let κ be a Suslin cardinal, and $F : \mathcal{P}_{\omega_1}(\kappa) \to \lambda < \Theta$ be such that $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)$ $[\text{cof}(F(S)) > \omega]$. Then, $\text{cof}([F]_\nu) > \kappa$, where $[F]_\nu$ is the ordinal represented by F in the ultrapower with respect to the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\kappa)$.*

We require the following lemma, which borrows from the ideas of Becker-Kechris [4]. In fact, it abstracts the main idea of that paper. Recall P is our fixed κ-Suslin set providing codes for the ordinals $< \kappa$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be κ-Suslin and universal for the κ-Suslin subsets of ω^ω. For $z \in \omega^\omega$, we say z codes a set $R_z \subseteq \kappa$ if

$U_z \doteq \{w \colon U(z,w)\} \subseteq P$ and is invariant in the codes. That is, if $w_1, w_2 \in P$, then $U(z,w_1) \leftrightarrow U(z,w_2)$. In this case, $R_z = \{\alpha < \kappa \colon \exists w \in P\ U(z,w) \wedge |w| = \alpha\}$. By the coding lemma, every subset of κ is of the form R_z for some z.

Lemma 1.2. *There is a relation $W(S,z,\alpha) \subseteq \mathcal{P}_{\omega_1}(\kappa) \times \omega^\omega \times \kappa$ satisfying the following:*

(1) $W(S,z,\alpha) \Rightarrow \alpha \in S$.

(2) $\forall S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall \alpha \in S\ \{z : W(S,z,\alpha)\} \in \mathbf{\Sigma}^1_1$.

(3) Suppose $z \in \omega^\omega$ codes a set $R_z \subseteq \kappa$. Then for all honest $S \in \mathcal{P}_{\omega_1}(\kappa)$ and all $\alpha \in S$, $[W(S,z,\alpha) \Rightarrow R_z(\alpha)]$.

(4) If $z \in \omega^\omega$ codes a relation $R_z \subseteq \kappa$ and $R_z(\alpha)$, then $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ W(S,z,\alpha)$.

Proof. We follow Becker-Kechris [4]. Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for κ-Suslin relations on ω^ω. Let $S \subseteq (\omega \times \omega \times \kappa)^{<\omega}$ be a tree with $U = p[S]$. Let $T \subseteq (\omega \times \kappa)^{<\omega}$ be a tree such that $p[T] = P$, the set of codes of ordinals $< \kappa$. For $\eta < \kappa$, define T^η by:

$$((a_0, \dots, a_n), (\alpha_0, \dots, \alpha_n)) \in T^\eta \leftrightarrow ((a_0, \dots, a_n), (\alpha_0, \dots, \alpha_n)) \in T \wedge \alpha_0 \leq \eta.$$

Thus, $x \in p[T^\eta]$ iff $x \in P$ and $|x| \leq \eta$. Let $T^* \subseteq \bigcup_{n \in \omega} (\omega^{n+1} \times (\omega^\omega)^{n+1})$ be the code set for T, that is,

$$\begin{aligned}&((a_0, \dots, a_n), (w_0, \dots, w_n)) \in T^* \leftrightarrow \\ &\qquad\qquad w_0, \dots, w_n \in P \wedge ((a_0, \dots, a_n), (|w_0|, \dots, |w_n|)) \in T.\end{aligned}$$

By the coding lemma, T^* is κ-Suslin. Define $Q_n \subseteq \omega^\omega \times \omega^{n+1} \times (\omega^\omega)^{n+1}$ by:

$$\begin{aligned}Q_n(x, (a_0, \dots, a_n), (w_0, \dots, w_n)) &\leftrightarrow x \in P \wedge \forall i \leq n\ (w_i \in P) \wedge \\ &\quad ((a_0, \dots, a_n), (|w_0|, \dots, |w_n|)) \in T^{|x|} \\ &\leftrightarrow x \in P \wedge \forall i \leq n\ (w_i \in P) \wedge \\ &\quad ((a_0, \dots, a_n), (w_0, \dots, w_n)) \in T^* \wedge |w_0| \leq |x|.\end{aligned}$$

Clearly each Q_n is κ-Suslin. Let $S_n \subseteq (\omega \times \omega^{n+1} \times \omega^{n+1} \times \kappa)^{<\omega}$ be a tree such that $Q_n = p[S_n]$ (identifying integers a_i with reals in some reasonable manner).

We define $W(S,z,\alpha)$ iff $\alpha \in S$ and II has a winning strategy in the game $G(S,z,\alpha)$. The game $G(S,z,\alpha)$ is played as follows.

$$\begin{array}{llll}
\mathrm{I} & y(0) & y(1) & y(2)\dots \\
 & \eta_0 & \eta_1 & \eta_3\cdots \\
 & & & \\
\mathrm{II} & x(0) & x(1) & x(2)\dots \\
 & f(0) & f(1) & f(2)\dots \\
 & g(0) & g(1) & g(2)\dots \\
 & w_0(0) & w_0(1) & w_0(2)\dots \\
 & h_0(0) & h_0(1) & h_0(2)\dots \\
 & \vdots & & \\
 & w_k(0) & w_k(1) & w_k(2)\dots \\
 & h_k(0) & h_k(1) & h_k(2)\dots \\
 & \vdots & &
\end{array}$$

Here, $y(m)$, $x(m)$, $w_k(m) \in \omega$ and η_m, $f(m)$, $g(m)$, $h_k(m) \in S$. II must play out infinitely many w_k, h_k, and these moves are played in a reasonable order. Specifically, II does not have to begin playing w_k, h_k until I has played $y(0), \dots, y(k)$ and $\eta_0, \dots, \eta_k$.

We define now the rules of the game. The first player to violate one of the rules loses. Otherwise, II wins. The rules of the game $G(S, z, \alpha)$ are:

(a) $\eta_0 \leq \alpha$, $f(0) \leq \alpha$.

(b) $\forall l \in \omega\ ((y(0), \dots, y(l)), (\eta_0, \dots, \eta_l)) \in T$.

(c) $\forall l \in \omega\ ((x(0), \dots, x(l)), (f(0), \dots, f(l))) \in T$.

(d) $\forall l \in \omega\ ((z(0), \dots, z(l)), (x(0), \dots, x(l)), (g(0), \dots, g(l))) \in S$.

(e) $\forall n, l \in \omega\ ((x(0), \dots, x(l)), (y(0), \dots, y(n)), (w_0(0), \dots, w_0(l)), \dots, (w_n(0), \dots, w_n(l)), (h_n(0), \dots, h_n(l))) \in S_n$.

This completes the definition of $W(S, z, \alpha)$. We now verify properties (1)-(4). (1) is trivial, and (2) follows easily since for fixed S we may identify S with ω, all the ordinal moves of $G(S, z, \alpha)$ come from S, and the game $G(S, z, \alpha)$ is closed for II. To see (3), suppose S is honest, $z \in \omega^\omega$ codes a relation $R_z \subseteq \kappa$, $\alpha \in S$, and II wins $G(S, z, \alpha)$. Using honesty, let I play $y, \vec{\eta}$ such that $y \in P$, $|y| = \alpha$, $\vec{\eta} \in S^\omega$, $\eta_0 = \alpha$, and $(y, \vec{\eta}) \in [T]$. Let II's winning strategy respond with x, f, g, w_k, h_k. Then f witnesses $x \in P$ and $|x| \leq \alpha$. Also, g witnesses $R_z(|x|)$. Finally, the w_k, h_k witness that $y \in p[T^{|x|}]$,and thus $|y| \leq |x|$. Hence $|x| = \alpha$, and thus, $R_z(\alpha)$. (4) follows as in Becker-Kechris, using a normality argument. Specifically, suppose $z \in \omega^\omega$ codes a relation $R_z \subseteq \kappa$. By normality, almost all $S \in \mathcal{P}_{\omega_1}(\kappa)$ satisfy:

(i) S is honest.

(ii) $\forall \alpha \in S\ (R_z(\alpha) \Rightarrow \exists x \in P\ \exists f, g \in S^\omega\ |x| = \alpha \wedge f(0) = \alpha \wedge (x, f) \in [T] \wedge (z, x, g) \in [S])$. Furthermore, for this x we have:

(iii)

$$\forall y(0),\dots,y(n) \in \omega\ \forall \eta_0,\dots,\eta_n \in S\ \Big\{((y(0),\dots,y(n)),(\eta_0,\dots,\eta_n)) \in T^{|x|} \Rightarrow \exists w_0,\dots,w_n \in P\ \exists h_n \in S^\omega\ \forall i \le n(|w_i| = \eta_i) \wedge (x,(y(0),\dots,y(n)),(w_0,\dots,w_n),h_n) \in [S_n]\Big\}.$$

For S satisfying (i)-(iii), and $\alpha \in S$ with $R_z(\alpha)$, it is easy to see that II wins $G(S,z,\alpha)$, and thus $W(S,z,\alpha)$. □

We will also use minor variations of W. For example, for z coding relations $R_z \subseteq \kappa^{<\omega} \times \kappa \times \omega$ (*e.g.*, strategies), there is a corresponding relation $W(S,z,\vec{\alpha},\beta,m)$. These relations are fixed for the rest of the paper.

Definition 1.1. Suppose $z \in \omega^\omega$ codes a strategy (Lipschitz continuous function) from κ^ω to $\kappa^\omega \times \omega^\omega$, and $S \in \mathcal{P}_{\omega_1}(\kappa)$. We say S is *strongly z-acceptable* if $\forall \vec{\alpha} \in S^{<\omega}\ \exists \beta \in S\ \exists m \in \omega\ W(S,z,\vec{\alpha},\beta,m)$.

From the properties of W and normality, we easily have the following:

(1) $\forall S \in \mathcal{P}_{\omega_1}(\kappa)\ \{z : S \text{ is strongly } z\text{-acceptable }\} \in \mathbf{\Sigma}^1_1$.

(2) If $S \in \mathcal{P}_{\omega_1}(\kappa)$ is honest and strongly z-acceptable, and z codes a strategy from $\kappa^\omega \to \kappa^\omega \times \omega^\omega$, then S is closed under the strategy coded by z.

(3) If z codes a strategy from $\kappa^\omega \to \kappa^\omega \times \omega^\omega$, then $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)$ S is strongly z-acceptable.

Proof of theorem 1.2. Suppose towards a contradiction that $h : \kappa \to [F]_\nu$ is cofinal. Consider the following game:

I	α_0		α_1		$\alpha_2 \dots$
II		$\beta_0, u(0)$		$\beta_1, u(1) \dots$	

Here I, II play ordinals $\alpha_i, \beta_i < \kappa$, and II plays integers $u(i)$ as well, building $u \in \omega^\omega$. II wins iff u codes a $\mathbf{\Sigma}^1_1$-bounded prewellordering (C_u, ϕ_u) of length $|u|$, say, and $R(G(\alpha_0,\beta_0,\alpha_1,\beta_1,\dots),u)$, where $R \subseteq \omega^\omega \times \omega^\omega$ is defined by: $R(x,u) \leftrightarrow F(S) = |u|$, where $S = \{|x_0|,|x_1|,\dots\}$. By lemma 1.1, this game is determined. Also, I cannot win, for if I won by σ, let S be an honest set closed under σ (and integer moves by II). II then defeats σ by enumerating S and playing u such that $|u| = F(S)$. Fix then a winning strategy $\mathcal{F}$ for II, which we also regard as a continuous function from $\kappa^\omega \to \omega^\omega$ by ignoring II's ordinal moves. We thus have that $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa) \forall^* s \in S^\omega\ \mathcal{F}(s)$ codes an ordinal, and $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa) \forall^* s\ |\mathcal{F}(s)| = F(S)$.

If $\lambda < [F]_\nu$, we say a real z is λ-good if:

(1) z codes a strategy (i.e., a Lipschitz continuous function) from κ^ω to $\kappa^\omega \times \omega^\omega$ such that if $S \in \mathcal{P}_{\omega_1}(\kappa)$ is honest and closed under $z, \mathcal{F}$, and s_0 enumerates S, then $z(s_0)$ (= the real part of II's response against s_0) codes a comeager set $A_{z(s_0)} \subseteq S^\omega$, and a continuous function $z(s_0,-) : A_{z(s_0)} \to \omega^\omega$ such that for all $s_1 \in A_{z(s_0)}$, $z(s_0,s_1) = w$ is in the $\mathbf{\Sigma}^1_1$-bounded union (C_u,ϕ_u) coded by $u = \mathcal{F}(s_1)$.

(2) $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^* s_0, s_1 \in S^\omega$ the rank of $w = z(s_0,s_1)$ in the $\mathbf{\Sigma}^1_1$-bounded union coded by $u = \mathcal{F}(s_1)$ is $> \lambda(S)$.

We first claim that $\forall \lambda < [F]_\nu\ \exists z \in \omega^\omega\ z$ is λ- good. To see this, fix a function $S \to \lambda(S)$ representing λ with respect to ν, and play the game where I plays α_i, II plays

$\beta_i, z(i)$, and II wins iff z codes a comeager set $A_z \subseteq S^\omega$, where $S = \{|r_0|, |r_1|, \dots\}$, $r = G(\alpha_0, \beta_0, \alpha_1, \beta_1, \dots)$, and a continuous function $z(-) : A_z \to \omega^\omega$ such that for all $s_1 \in A_z$, $z(s_1) = \overline{0}$ if $\mathcal{F}(s_1) = u$ does not code $F(S)$, and otherwise the rank of $z(s_1)$ in the $\mathbf{\Sigma}^1_1$-bounded union coded by u is $> \lambda(S)$. This game is again determined by lemma 1.1. Suppose I won by σ. Let S be an honest set closed under σ, $\mathcal{F}$ such that $\mathrm{cof}(F(S)) > \omega$, and $F(S) > \lambda(S)$. II will enumerate S in the β's. Let $R \subseteq S^\omega \times \omega^\omega$ be defined by: $R(s_1, w) \leftrightarrow [s_1 \in S^\omega \wedge s_1$ enumerates $S \wedge w$ is in the $\mathbf{\Sigma}^1_1$-bounded union coded by $\mathcal{F}(s_1)$ and has rank $\geq \lambda(S)]$. From AD, we may uniformize R by R' on a comeager set. Also, every function defined on a comeager subset of S^ω is continuous restricted to a comeager set. Let $z \in \omega^\omega$ code such a comeager set $A_z \subseteq S^\omega$ (coding neighborhoods using only the β_i; see our previous remarks) and continuous function $z(-) : A_z \to \omega^\omega$. If II plays this z, then II defeats I. A winning strategy τ for II then gives a λ-good real.

By the usual game argument, there is a Lipschitz continuous function $R : \kappa^\omega \to \omega^\omega$ such that for all $t \in \kappa^\omega$, $R(t)$ is η-good for some $\eta < [F]_\nu$, and $\forall \alpha < \kappa\, \forall T$ honest, closed under R, and containing α, and all t enumerating T, $R(\alpha^\frown t)$ is $h(\alpha)$-good.

For S honest, closed under $\mathcal{F}$, and $F(S)$ of cofinality $> \omega$, we define $G(S) \leq F(S)$ to be the least $\gamma \leq F(S)$ such that $\forall^* s_1 \in S^\omega\; G(S, s_1) < \gamma$. Also, $G(S, s_1) = \sup\{\phi_u(w) : u = \mathcal{F}(s_1), w \in B(S, s_1)\}$. Finally, $B(S, s_1) \subseteq \omega^\omega$ is defined by: $w \in B(S, s_1)$ iff $\exists \alpha \in S\; \exists t \in S^\omega\; \exists z \in \omega^\omega\; \exists s_0$ enumerating S $[z = R(\alpha^\frown t)$, S is strongly z-acceptable, $s_1 \in A_{z(s_0)}$, and $w = z(s_0, s_1)]$. Note that $B(S, s_1) \subseteq C_u$, where $u = \mathcal{F}(s_1)$, using property (3) of W. A straightforward computation, using property (2) of W, also shows that $B(S, s_1) \in \mathbf{\Sigma}^1_1$. By boundedness, $G(S, s_1) < F(S)$. Since $\mathrm{cof}(F(S)) > \omega$, it then follows that $G(S) < F(S)$.

Fix $\alpha < \kappa$. We show that $[G]_\nu \geq h(\alpha)$, a contradiction. Fix $t \in \kappa^\omega$ enumerating an honest set closed under R, containing α, and let $z = R(\alpha^\frown t)$. Suppose $[G]_\nu < h(\alpha)$, towards a contradiction. Fix a function $S \to h(\alpha)(S)$ representing $h(\alpha)$. Fix S honest, containing α and range(t), closed under $\mathcal{F}$, with S strongly z-acceptable, $F(S) > h(\alpha)(S) > G(S)$, $\mathrm{cof}(F(S)) > \omega$, and such that $\forall^* s_0, s_1 \in S^\omega\; \phi_u(w) \geq h(\alpha)(S)$ where $u = \mathcal{F}(s_1)$, $s_1 \in A_{z(s_0)}$, $w = z(s_0, s_1)$. Let s_0 enumerate S and such that $\forall^* s_1 \in S^\omega\; \phi_u(w) \geq h(\alpha)(S)$, u, w as above. From the definition of $G(S)$ we have $\forall^* s_1 \in S^\omega\; G(S, s_1) < h(\alpha)(S)$, and, using s_0, z as witnesses, we have also $\forall^* s_1 \in S^\omega\; w = z(s_0, s_1) \in B(S, s_1)$. Let s_1 be in the intersection of these three comeager sets, and let $w = z(s_0, s_1) \in B(S, s_1)$. Then $G(S, s_1) < h(\alpha)(S)$, but also $\phi_u(w) \geq h(\alpha)(S)$, and so $G(S, s_1) \geq h(\alpha)(S)$ as well, a contradiction.

□

Remark 1.1. The observation that $\nu(A) = 1$ iff there is a strategy $\tau : \kappa^\omega \to \kappa^\omega$ such that A contains all honest sets closed under τ, together with the fact that any normal, fine ultrafilter on $\mathcal{P}_{\omega_1}(\kappa)$ must contain all such sets, gives another proof of Woodin's theorem on the uniqueness of ν in the case κ a Suslin cardinal. This proof is due to Becker (and predates Woodin's result).

Remark 1.2. The converse of theorem 1.2 is also true, that is, if $\forall^* S\; \mathrm{cof}(F(S)) = \omega$, then $\mathrm{cof}([F]_\nu) \leq \kappa$. To see this, fix a Lipschitz continuous F' such that $\forall^* S \forall^* s\; F'(s)$ codes $F(S)$, that is, $F(S) = \sup\{|F'(s)_i| : i \in \omega\}$. Define for S honest, closed under F', $G(S) = \{G(S)(p, i) : p \in S^{<\omega}, i \in \omega\}$, where $G(S)(p, i) =$ the least $\beta < F(S)$ (if one exists, otherwise set $G(S)(p, i) = 0$) such that $\forall^*_p s \in S^\omega\; |F'(s)_i| < \beta$. Then $[G]_\nu$ has size $\leq \kappa$, and is cofinal in $[F]_\nu$.

Remark 1.3. Theorem 1.2, and in fact our main result, theorem 2.1, are also valid under the assumption AD + every set of reals has a scale. This is because lemma 1.1 is true in this context (the game G_A in this case is now directly Suslin and co-Suslin). It is not clear, however, if theorem 1.2 (or theorem 2.1) holds just assuming AD. See also remark 2.1

Finally, we note the following easy consequence of theorem 1.1, which we will use later.

Lemma 1.3. *Let κ be a Suslin cardinal, and $\lambda < \Theta$ with $\mathrm{cof}(\lambda) > \kappa$. Suppose $f\colon \mathcal{P}_{\omega_1}(\kappa) \to \lambda$. Then there is a $\lambda' < \lambda$ such that $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ f(S) < \lambda'$.*

Proof. By theorem 1.1, fix a κ-Suslin bounded prewellordering (C, ϕ) of length λ. The usual game argument gives a Lipschitz continuous function $F\colon \kappa^\omega \to \omega^\omega$ such that $\forall s \in \kappa^\omega\ F(s) \in C$, and $\forall^* S\ \forall^* s \in S^\omega\ \phi(F(s)) > f(S)$. Let $D = \{F(s) \colon s \in \kappa^\omega\}$, so $D \subseteq C$, and easily D is κ-Suslin. Let $\lambda' = \sup\{\phi(z) \colon z \in D\}$. □

2. The Main Result

Recall we assume throughout $AD + V = L(\mathbb{R})$. Throughout this section, κ will denote a Suslin cardinal. Throughout, let $\mathcal{M}$ denote the transitive collapse of the ultrapower of $L(\mathbb{R})$ by the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\kappa)$.

The next theorem is our main result.

Theorem 2.1. *Let κ be a Suslin cardinal, and $\lambda < \Theta$. Suppose $f\colon \kappa \to \lambda$ and $\forall \alpha < \kappa\, \mathrm{cof}(f(\alpha)) \leq \kappa$. Then there is a set $A \in \mathcal{M} \subseteq L(\mathbb{R})$, $|A| \leq \kappa$ such that $\mathrm{range}(f) \subseteq A$ and $A \cap f(\alpha)$ is cofinal in $f(\alpha)$ for all $\alpha < \kappa$.*

As an immediate corollary we have:

Corollary 2.1. *$\boxminus_{\kappa,\lambda}$ holds for all Suslin cardinals κ, and all $\lambda < \Theta$.*

Remark 2.1. The hypothesis $V = L(\mathbb{R})$ can be relaxed in several ways. Assuming just AD, $\boxminus_{\kappa,\lambda}$ holds for all $\lambda < \Theta^{L(\mathbb{R})}$ (and κ a Suslin cardinal always). This follows easily from the coding lemma, which implies that V (our model of $ZF + AD$) and $L(\mathbb{R})$ have the same subsets of λ. Secondly, as we pointed out in remark 1.3, $\boxminus_{\kappa,\lambda}$ holds for all $\lambda < \Theta^V$ if $V \models ZF + AD +$"every set of reals has a scale". Finally, if there is an outer model $V' \supseteq V$ of $ZF + AD$ with $V' \cap \omega^\omega = V \cap \omega^\omega$ and with Θ^V within scales of V' (*i.e.*, there is a scaled pointclass $\mathbf{\Gamma}$ in V' with $o(\mathbf{\Delta}) > \Theta^V$), then $\boxminus_{\kappa,\lambda}$ holds in V for all $\lambda < \Theta^V$. This follows using the same proof given below, working in V. At those points where we need the determinacy of an ordinal game of the type described in lemma 1.1, we use the fact that the game is Suslin, co-Suslin in V', and hence determined in V' (the set of reals A, in the terminology of lemma 1.1, will be in V, so $A \in \mathbf{\Delta}$). However, by the coding lemma, any winning strategy must lie in V (the coding lemma implies V, V' have the same subsets of $\kappa < \Theta^V$, using the fact that $V \cap \omega^\omega = V' \cap \omega^\omega$).

We turn now to the proof of theorem 2.1. We require two lemmas. The first covers the "low cofinality" case, and the second the "high cofinality" case.

Lemma 2.1. *Suppose $A \subseteq \lambda < \Theta$, $A \in \mathcal{M}$, $|A| \leq \kappa$, and $\forall \alpha \in A\ \mathrm{cof}(\alpha) \leq \kappa$. Then there is a $B \in \mathcal{M}$ such that $B \supseteq A$, $|B| \leq \kappa$, and $\forall \alpha \in A\ B \cap \alpha$ is cofinal in α.*

Proof. Let $[F]_\nu = A$, and we may assume $\forall S \in \mathcal{P}_{\omega_1}(\kappa)$ $F(S)$ is a countable set of ordinals. We may also assume (by theorem 1.2) that $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)$ $\forall \alpha \in F(S)$ $\mathrm{cof}(\alpha) \leq \omega$. By countable choice, $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)$ $\exists B \subseteq \sup(F(S))$, B countable, such that $\forall \alpha \in F(S)$ $B \cap \alpha$ is cofinal in α. For such S, there then is also a function $h : \omega \to \sup(F(S))$ such that $\forall \alpha \in F(S)$ $\{h(n) \colon h(n) < \alpha\}$ is cofinal in α. By the usual game argument, there is a Lipschitz continuous function $F' \colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that for all honest S closed under F', and any s enumerating S, $F'(s)$ codes a function (which we also denote by $F'(s)$) from ω to $\sup(F(S))$ such that $\forall \alpha \in F(S)$ $\{F'(s)(n) \colon F'(S)(n) < \alpha\}$ is cofinal in α. We define now G such that $\forall^* S$ $G(S)$ is a function from $\omega \times S^{<\omega}$ to $\sup(F(S))$ such that $\forall \alpha \in F(S)$ $\{G(S)(n,p) \colon n \in \omega,\ p \in S^{<\omega}\} \cap \alpha$ is cofinal in α. Define $G(n,p) =$ the least γ such that $\forall^*_p s \in S^\omega$ $F'(s)(n) < \gamma$, if one exists, and otherwise $= 0$. For S honest, closed under F', $\alpha \in F(S)$, and $\beta < \alpha$, by additivity of category it follows that $\exists p \in S^{<\omega}$ $\exists n \in \omega$ $\exists \beta' < \alpha$ $[\beta' \geq \beta \wedge \forall^*_p s \in S^\omega\ F'(s)(n) = \beta']$. Then, $\beta \leq G(S)(n,p) < \alpha$. G is as required. □

Lemma 2.2. *Suppose $A \subseteq \lambda < \Theta$, $A \in \mathcal{M}$, $|A| \leq \kappa$, and $\forall \alpha \in A$ $\mathrm{cof}(\alpha) > \kappa$. Suppose $C \subseteq \mathrm{ON}$, $|C| \leq \kappa$. Then there is a $B \in \mathcal{M}$ such that $B \supseteq A$, $|B| \leq \kappa$, and $\forall \gamma \in C$ $\forall \alpha \in A$ $\exists \beta \in B$ $(\beta < \alpha \wedge \gamma \notin [\beta, \alpha))$.*

Proof. Let $[F]_\nu = A$, and we may assume that $\forall S$ $F(S)$ is a countable set of ordinals, each of cofinality $\geq \omega_1$. Let $F' : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ be a Lipschitz continuous function such that for all S honest and closed under F', and s enumerating S, $F'(s)$ codes ω many reals $F'(s)_i$, each of which codes an ordinal of uncountable cofinality (by coding a $\mathbf{\Sigma}^1_1$-bounded prewellordering of that length), and $F(S) = \{|F'(s)_i| : i \in \omega\}$.

For γ an ordinal such that $\forall^* S$ $\gamma(S) < \sup(F(S))$, that is, $\gamma < \sup(A)$, we say a real z is γ-good if z codes a Lipschitz continuous function from κ^ω to $\kappa^\omega \times \omega^\omega$ satisfying:

(1) for all S honest, closed under z, and s_0 enumerating S, $z(s_0)$ codes a comeager $A_{z(s_0)} \subseteq S^\omega$ and a continuous function $z(s_0)(-) : A_{z(s_0)} \to \omega^\omega$ such that $\forall s_1 \in A$ $z(s_0)(s_1) = \langle n, w \rangle$ where w is in the $\mathbf{\Sigma}^1_1$-bounded union coded by $u = F'(s_1)_n$.

(2) $\forall^* S$ $\forall^* s_0$ $\forall^* s_1 \in S^\omega$ $[\sup\{|F(s_1)_m| : |F(s_1)_m| < |F(s_1)_n|\} \leq \gamma(S) < |F(s_1)_n|]$, where $n = z(s_0)(s_1)_0$.

(3) $\forall^* S$ $\forall^* s_0$ $\forall^* s_1 \in S^\omega$ $\phi_u(w) \geq \gamma(S)$, where $w = z(s_0)(s_1)_1$, $u = F'(s_1)_n$, and $n = z(s_0)(s_1)_0$.

For each $\gamma \in C$, $\gamma < \sup(A)$, the usual game argument shows that there is a z which is γ-good. Let R be a Lipschitz continuous function $R \colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that for all $t \in \kappa^\omega$, $R(t)$ is η-good for some $\eta < \sup(A)$, and $\forall \gamma < \kappa$ $\forall T$ honest, closed under R, and containing γ, and all t enumerating T, $R(\gamma^\frown t)$ is $h(\gamma)$-good, where $h \colon \kappa \to C \cap \sup(A)$ is onto.

For S honest, closed under F', we define $G(S)$. We will then let $B = [G]_\nu$. We define for $\alpha \in F(S)$ an ordinal $G(S)(\alpha) < \alpha$, and then set $G(S) = \{G(S)(\alpha) \colon \alpha \in F(S)\}$. We set $G(S)(\alpha) =$ the maximum of $\sup\{\alpha' \in F(S) \colon \alpha' < \alpha\} + 1$ and the least $\beta < \alpha$ such that $\forall^* s_1 \in S^\omega$ $\forall n \in \omega$ $G(S)(\alpha, s_1, n) < \beta$. We define $G(S)(\alpha, s_1, n)$ as follows. If $|F'(s_1)_n| \neq \alpha$, set $G(S)(\alpha, s_1, n) = 0$. Otherwise, set $G(S)(\alpha, s_1, n) = \sup\{\phi_u(w) : u = F'(s_1)_n, w \in A(\alpha, s_1, n)\}$, where $w \in A(\alpha, s_1, n)$ iff $\exists z \in \omega^\omega$ $\exists \tau \in S^\omega$ $\exists s_0 \in S^\omega$ $[z = R(\tau)$, S is strongly z-acceptable, s_0 enumerates S, $s_1 \in A_{z(s_0)}$, $n = z(s_0)(s_1)_0$, and $w = z(s_0)(s_1)_1]$.

Since $A(\alpha, s_1, n) \in \mathbf{\Sigma}^1_1$, $G(S)(\alpha, s_1, n) < \alpha = |F'(s_1)_n|$ by boundedness. It follows that $G(S)(\alpha) < \alpha$.

To see this works, fix $\gamma \in C$, $\gamma < \sup(A)$, and fix $\tau \in \kappa^\omega$ such that $z = R(\tau)$ is γ-good. Fix $S \to \gamma(S)$ representing γ. Suppose towards a contradiction that $H \colon \mathcal{P}_{\omega_1}(\kappa) \to \mathrm{ON}$ is such that $\forall^* S\ H(S) \in F(S)$ and $\forall^* S\ G(S)(H(S)) \leq \gamma(S) < H(S)$. Let S be honest, closed under F', S strongly z-acceptable, $G(S)(H(S)) \leq \gamma(S) < H(S)$, and such that $\forall^* s_0, s_1 \in S^\omega\ \phi_u(w) \geq \gamma(S)$, where $u = F'(s_1)_n$, $w = z(s_0)(s_1)_1$, and $n = z(s_0)(s_1)_0$, and also $|F'(s_1)_n| = H(S)$. Let s_0 enumerate S and such that $\forall^* s_1\ \forall n\ G(S)(H(S), s_1, n) < G(S)(H(S))$, $\forall^* s_1\ \phi_u(w) \geq \gamma(S)$, with u, w as above, and $\forall^* s_1\ |F'(s_1)_n| = H(S)$ where again $n = z(s_0)(s_1)_0$. If we fix $s_1 \in S^\omega$ in the intersection of these three comeager sets, and let $u = F'(s_1)_n$, $w = z(s_0)(s_1)_1$, and $n = z(s_0)(s_1)_0$, then we have: $G(S)(H(S), s_1, n) < G(S)(H(S)) \leq \gamma(S) \leq \phi_u(w) \leq G(S)(H(S), s_1, n)$, a contradiction (the last inequality comes from the definition of $G(S)(H(S), s_1, n)$, using these s_0, z as witnesses). $\square$

Proof of theorem 2.1. Let $f \colon \kappa \to \lambda < \Theta$, and $C = \mathrm{range}(f)$. Let $A_0 = \{\sup(C)\}$. In general, suppose that after step n we have defined a set $A_n \in \mathcal{M}$ with $|A| \leq \kappa$. Let $D_n = \{\alpha \in A_n \colon \mathrm{cof}(\alpha) \leq \kappa\}$, and $E_n = \{\alpha \in A_n \colon \mathrm{cof}(\alpha) > \kappa\}$. Note that D_n, E_n are in $\mathcal{M}$ as well (by theorem 1.2 and remark 1.2). Applying lemmas 2.1, 2.2 to D_n, E_n respectively produces D'_n, $E'_n \in \mathcal{M}$ of size $\leq \kappa$ (using $C = \mathrm{range}(f)$ in lemma 2.2). Let $A_{n+1} = D'_n \cup E'_n$. Continuing, using DC, we define A_n for all $n \in \omega$. Note that $A_0 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \ldots$. Let $A = \bigcup_{n \in \omega} A_n$. Thus, $|A| \leq \kappa$, and $A \in \mathcal{M}$ (as $\mathcal{M}$ is closed under ω sequences).

First note that $C \subseteq A$. To see this, let $\gamma \in C$, and let $\alpha \in A$ be least such that $\gamma \leq \alpha$. Say $\alpha \in A_n$. If $\gamma < \alpha$, then from lemma 2.1, 2.2 and the definitions of D'_{n+1}, E'_{n+1}, there is a $\beta \in A_{n+1} \subseteq A$ such that $\gamma \leq \beta < \alpha$, a contradiction.

Finally, let $\gamma \in C$. Let n be such that $\gamma \in A_n$. Since $\mathrm{cof}(\gamma) \leq \kappa$, from lemma 2.1 it follows that $A_{n+1} \cap \gamma$ is cofinal in γ. Thus, $\forall \gamma \in C\ A \cap \gamma$ is cofinal in γ. $\square$

3. On λ-Regular Cardinals

In this section we use theorem 2.1 to show that for κ a regular cardinal which is either a Suslin cardinal or the successor of a Suslin cardinal, κ is $\boldsymbol{\delta}^2_1$-supercompact. Recall we are assuming $AD + V = L(\mathbb{R})$ throughout.

We recall a definition of Becker [3].

Definition 3.1. A cardinal κ is λ-regular if for every $f \colon \alpha \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$, where $\alpha < \kappa$, there is an $A \subseteq \lambda$, $|A| < \kappa$, such that $\forall \alpha' < \alpha\ \exists B \in f(\alpha')\ [B \subseteq A]$.

Becker (see [3]) showed that if κ is regular and either a Suslin cardinal or the successor of a Suslin cardinal, then $\forall \lambda \leq \boldsymbol{\delta}^2_1$, κ is λ-regular iff κ is λ-supercompact.

Note also that for all κ, λ, if κ^+ is λ-regular, then $\boxminus_{\kappa,\lambda}$ holds (apply λ-regularity to the function f with domain κ such that for all $\alpha < \kappa$, $f(\alpha)$ is the collection of all $B \in \mathcal{P}_{\kappa^+}(\lambda)$ which are cofinal in $f(\alpha)$).

We have now:

Theorem 3.1. *Let κ be regular and either a Suslin cardinal or the successor of a Suslin cardinal. Then for all $\lambda < \Theta$, κ is λ-regular.*

Corollary 3.1. *Let κ be regular and either a Suslin cardinal or the successor of a Suslin cardinal. Then κ is $\boldsymbol{\delta}^2_1$-supercompact.*

Corollary 3.2. *The projective ordinals $\boldsymbol{\delta}^1_n$ are all $\boldsymbol{\delta}^2_1$-supercompact.*

Remark 3.1. Theorem 3.1 and corollaries 3.1, 3.2 are valid assuming just AD, provided we replace Θ and $\boldsymbol{\delta}^2_1$ with $\Theta^{L(\mathbb{R})}$ and $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$. For suppose $\kappa < \lambda < \Theta^{L(\mathbb{R})}$, and let $f\colon \alpha \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$, where $\alpha < \kappa$. Note that κ also satisfies the hypothesis of theorem 3.1 in $L(\mathbb{R})$, since (by the coding lemma) V and $L(\mathbb{R})$ have the same Suslin cardinals below $\Theta^{L(\mathbb{R})}$. There is a prewellordering in $L(\mathbb{R})$ of length α. Hence, by the coding lemma, there is a function $f' \in L(\mathbb{R})$ such that $f'(\beta) \subseteq f(\beta)$ and $f'(\beta) \neq \emptyset$ for all $\beta < \alpha$. Applying theorem 3.1 in $L(\mathbb{R})$ to f' produces the desired covering set.

We require the following lemma, which uses again the technique of the previous results, and is similar to lemma 2.2.

Lemma 3.1. *Let $\kappa \leq \lambda < \Theta$, κ a Suslin cardinal. Let $F\colon \kappa^\omega \to \omega^\omega$ be Lipschitz continuous such that $\forall T$ honest, closed under F, and all $\beta \in T$ and t enumerating T, $F(\beta^\frown t)$ codes a set $B_{F(\beta^\frown t)} \subseteq \lambda$ with $|B_{F(\beta^\frown t)}| \leq \kappa$. Let $b\colon \kappa \to \lambda$ be such that $\forall \alpha < \kappa\ \operatorname{cof}(b(\alpha)) > \kappa$. Then there is a $g\colon \kappa \to \lambda$ with $g(\alpha) < b(\alpha)$ for all α, and such that $\forall \beta < \kappa\ \forall^* T \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^* t \in T^\omega\ \forall \alpha < \kappa\ [B_{F(\beta^\frown t)} \cap b(\alpha) \subseteq g(\alpha)]$.*

Proof. By the usual game argument, there is a Lipschitz continuous $R\colon \kappa^\omega \to \omega^\omega$ such that $\forall \alpha < \kappa$, $\forall S$ honest, closed under R and containing α, and all s enumerating S, $u = R(\alpha^\frown s)$ codes a κ-Suslin bounded prewellordering (C_u, ϕ_u) of length $b(\alpha)$.

If $h\colon \kappa \to \lambda$ is such that $\forall \alpha < \kappa\ h(\alpha) < b(\alpha)$, we say $z \in \omega^\omega$ is h-good provided z codes a strategy on κ, and if $\alpha < \kappa$, S is honest, closed under z, R (that is, the strategy coded by z), and contains α, and s_0 enumerates S, then $z(\alpha^\frown s_0)$ codes a comeager set $A^z_{(\alpha,s_0)} \subseteq S^\omega$ and a continuous function $f^z_{(\alpha,s_0)} : A^z_{(\alpha,s_0)} \to \omega^\omega$ such that for all $s_1 \in A^z_{(\alpha,s_0)}$, $w = f^z_{(\alpha,s_0)}(s_1)$ is in the κ-Suslin bounded union coded by $u = R(\alpha^\frown s_1)$, and $\phi_u(w) = h(\alpha)$. We say z is good if it is good for some h. Note that if z is good, then the corresponding function h_z is uniquely determined.

The usual game argument gives that for any such h, there is an h-good real z.

The usual game argument also gives a strategy $\mathcal{T}$ on κ such that for all $\beta < \kappa$, and all honest T closed under $\mathcal{T}$ and containing β, and t_0 enumerating T, $\mathcal{T}(\beta^\frown t_0)$ codes a comeager set A_{β,t_0} and a continuous function $f_{\beta,t_0} : A_{\beta,t_0} \to \omega^\omega$ such that for all $t_1 \in A_{\beta,t_0}$, $f_{\beta,t_0}(t_1)$ is h-good, where $h(\alpha) = \sup(b(\alpha) \cap B_{F(\beta^\frown t_1)})$.

Define $D \subseteq \omega^\omega$ by: $z \in D \leftrightarrow \exists \beta < \kappa\ \exists T \in \mathcal{P}_{\omega_1}(\kappa)\ \exists t_0, t_1 \in T^\omega$ [T is honest, closed under $\mathcal{T}$, and contains β, t_0 enumerates T, $t_1 \in A_{\beta,t_0}$, and $z = f_{\beta,t_0}(t_1)$]. Easily, D is κ-Suslin. Also, $\forall \beta < \kappa\ \forall^* T \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^* t_1 \in T^\omega\ \exists z \in D$ [z is h good], where $h(\alpha) = \sup(b(\alpha) \cap B_{F(\beta^\frown t_1)})$. Also, every $z \in D$ is good.

For $\alpha < \kappa$, we now define $g(\alpha)$. First, for S honest, containing α, closed under R, and s_1 enumerating S, we define $g(\alpha, S, s_1)$. Let $u = R(\alpha^\frown s_1)$, so u codes a κ-Suslin bounded union (C_u, ϕ_u) of length $b(\alpha)$. Let $w \in A(\alpha, S, s_1) \leftrightarrow \exists z \in D$ $\exists s_0 \in S^\omega$ such that: S is closed under (the strategy coded by) z, s_0 enumerates S, $s_1 \in A^z_{\alpha,s_0}$, and $w = f^z_{\alpha,s_0}(s_1)$. Easily $A(\alpha, S, s_1)$ is κ-Suslin, and $A(\alpha, S, s_1) \subseteq C_u$. Let then $g(\alpha, S, s_1) = \sup\{\phi_u(w)\colon w \in A(\alpha, S, s_1)\}$, where $u = R(\alpha^\frown s_1)$. Thus, $g(\alpha, S, s_1) < b(\alpha)$. Since $\operatorname{cof} b(\alpha) > \omega$, let $g(\alpha, S) =$ the least $\gamma < b(\alpha)$ such that $\forall^* s_1 \in S^\omega\ g(\alpha, S, s_1) < \gamma$, defined for S honest containing α, closed under R. Finally, let $g(\alpha) =$ the least $\delta < b(\alpha)$ such that $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ g(\alpha, S) < \delta$. Since $\operatorname{cof}(b(\alpha)) > \kappa$, this is well-defined by lemma 1.3.

To show this works, it is sufficient to show that for any $z \in D$, $\forall \alpha < \kappa\ g(\alpha) > h_z(\alpha)$, where z is good for h_z. Fix $z \in D$, $\alpha < \kappa$. Let $S \in \mathcal{P}_{\omega_1}(\kappa)$ be honest, closed under R and the strategy coded by z, contain α, and be such that $g(\alpha, S) < g(\alpha)$. Let s_0 enumerate S and such that $\forall^* s_1 \in S^\omega\ [g(\alpha, S, s_1) < g(\alpha, S) \wedge s_1 \in A^z_{\alpha,s_0}]$. If we pick s_1 in these two comeager sets, we then have: (1) $g(\alpha, S, s_1) < g(\alpha)$, and (2) $w = f^z_{\alpha,s_0}(s_1) \in A(\alpha, S, s_1)$. From (2) it follows that $g(\alpha, S, s_1) \geq h_z(\alpha)$, and we are done. □

We turn now to the proof of theorem 3.1. We consider first the case κ the successor of a Suslin cardinal. So assume $\kappa = \bar\kappa^+$, where $\bar\kappa$ is a Suslin cardinal. By theorem 2.1, $\boxminus_{\bar\kappa,\lambda}$ holds for all $\lambda < \Theta$. Let $\lambda < \Theta$, and let $f\colon \bar\kappa \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$. We must find $A \subseteq \lambda$, $|A| \leq \bar\kappa$, such that $\forall \alpha < \bar\kappa\ \exists B \in f(\alpha)\ B \subseteq A$. Let $F\colon \bar\kappa^\omega \to \omega^\omega$ be Lipschitz continuous such that $\forall \alpha < \kappa\ \forall S$ honest, containing α, and closed under F $\forall s$ enumerating S, $F(\alpha^\frown s)$ codes a set $B_{F(\alpha^\frown s)} \in f(\alpha)$. We may assume λ is a minimal counterexample, and it follows easily that $\mathrm{cof}(\lambda) \leq \bar\kappa$. [If $\mathrm{cof}(\lambda) > \bar\kappa$, then there would be a $\lambda' < \lambda$ such that for all $\alpha < \bar\kappa$ there is some $B \in f(\alpha)$ with $B \subseteq \lambda'$. Apply induction to $f'(\alpha) = f(\alpha) \cap \{B\colon B \subseteq \lambda'\}$.] Let $A_0 \subseteq \lambda$, $|A_0| = \bar\kappa$ be cofinal in λ. Let B_0 be the elements of A_0 whose cofinality is $\leq \bar\kappa$, and C_0 those of cofinality $> \bar\kappa$. Let f_0, $g_0\colon \bar\kappa \to \lambda$ enumerate B_0, C_0 respectively. By $\boxminus_{\bar\kappa,\lambda}$, let $B_1 \subseteq \lambda$, $|B_1| \leq \bar\kappa$ be such that $\forall \alpha \in B_0$ $(B_1 \cap f(\alpha))$ is cofinal in $f(\alpha)$. Let $f_1\colon \bar\kappa \to B_1$ enumerate B_1 (allowing repetitions). By lemma 3.1, there is a $g_1\colon \bar\kappa \to \mathrm{ON}$ such that $g_1(\beta) < g_0(\beta)$ for all $\beta < \bar\kappa$, and such that

$$\forall \alpha < \bar\kappa\ \forall^* S \in \mathcal{P}_{\omega_1}(\bar\kappa)\ \forall^* s \in S^\omega\ \forall \beta < \bar\kappa\ B_{F(\alpha^\frown s)} \cap (g_1(\beta), g_0(\beta)) = \emptyset.$$

Let C_1 be the range of g_1, and let $A_1 = A_0 \cup B_1 \cup C_1$. Using DC, we similarly define $A_{n+1} = A_n \cup B_{n+1} \cup C_{n+1}$, f_{n+1}, g_{n+1} starting from A_n instead of A_0. Let $A = \bigcup_{n\in\omega} A_n$. To see A works, fix $\alpha < \bar\kappa$, and we show in fact that $\forall^* S \in \mathcal{P}_{\omega_1}(\bar\kappa)$ $\forall^* s \in S^\omega\ B_{F(\alpha^\frown s)} \subseteq A$. Note that (by countable additivity)

$$\forall^* S \in \mathcal{P}_{\omega_1}(\bar\kappa)\ \forall^* s \in S^\omega\ \forall n \in \omega\ \forall \beta < \bar\kappa\ B_{F(\alpha^\frown s)} \cap (g_{n+1}(\beta), g_n(\beta)) = \emptyset.$$

However, if $B \subseteq \lambda$ is such that $\forall n \in \omega\ \forall \beta < \bar\kappa\ B \cap (g_{n+1}(\beta), g_n(\beta)) = \emptyset$, then it easy to see that $B \subseteq A$. Namely, if $\alpha \in B - A$, and if γ_n is the least element in A_n greater than α, then we would have $\gamma_0 > \gamma_1 > \dots$. This completes the proof for κ the successor of a Suslin cardinal.

Consider the case κ a regular Suslin cardinal. If κ is a successor cardinal, note that there is an ordinal $\kappa' < \kappa$ and $P' \subseteq P$ such that $\psi_0\colon P' \overset{onto}{\to} \kappa'$, $\psi_n\colon P' \to \kappa'$ for all n, and the prewellordering of P' induced by ψ_0 is κ'-Suslin. [We use the fact from the scale analysis [12] that there is a scaled pointclass $\mathbf{\Gamma}$ closed under $\forall^{\omega^\omega}$, countable unions and intersections such that κ is the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings. A simple boundedness and closure argument then produces P', κ'.] The previous argument then works, replacing $\bar\kappa$ by κ'. Alternatively, it follows from [12] that κ^- is actually a Suslin cardinal.

So, assume κ is an inaccessible Suslin cardinal. From the scale analysis of [12] it follows that κ is a limit of Suslin cardinals. Let again λ be a minimal counterexample, $\kappa_0 < \kappa$ a Suslin cardinal (without loss of generality), and $f\colon \kappa_0 \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$ witnessing the failure of λ-regularity. Again, $\mathrm{cof}(\lambda) \leq \kappa_0$. As before, let $F\colon {\kappa_0}^\omega \to \omega^\omega$ be Lipschitz continuous such that $\forall \alpha < \kappa_0\ \forall$ honest S containing α and closed under F $\forall s$ enumerating S [$F(\alpha^\frown s)$ codes a set $B_{F(\alpha^\frown s)} \in f(\alpha)$]. By lemma 1.3, there is a Suslin cardinal $\kappa' < \kappa$ such that $\forall \alpha < \kappa_0\ \forall^* S \in \mathcal{P}_{\omega_1}(\kappa_0)$

$\forall^* s \in S^\omega\ |B_{F(\alpha^\frown s)}| < \kappa'$. Thus, increasing κ_0 if necessary, we may assume that κ_0 is the successor of a Suslin cardinal, $\mathrm{cof}(\lambda) < \kappa_0$ and $f\colon \kappa_0^- \to \mathcal{P}(\mathcal{P}_{\kappa_0}(\lambda)) - \{\emptyset\}$. The argument of the first case now finishes. This completes the proof of theorem 3.1. Λ

Remark 3.2. The proof of theorem 3.1 actually showed the following stronger result: Let $\kappa < \lambda < \Theta$, κ a Suslin cardinal. Let $F\colon \kappa^\omega \to \omega^\omega$ be Lipschitz continuous such that $\forall \alpha < \kappa\ \forall$ honest S containing α and closed under F $\forall s$ enumerating S $[F(\alpha^\frown s)$ codes a set $B_{F(\alpha^\frown s)} \in \mathcal{P}_{\kappa^+}(\lambda)]$. Then there is an $A \subseteq \lambda$, $|A| \leq \kappa$, such that $\forall \alpha < \kappa$ $\forall^* S\ \forall^* s \in S^\omega\ B_{F(\alpha^\frown s)} \subseteq A$.

Remark 3.3. Using the previous remark, one may give a direct proof of corollary 3.1, that is, one that does not appeal to λ-regularity. For example, consider the case $\kappa = \bar{\kappa}^+$, $\bar{\kappa}$ a Suslin cardinal. Briefly, let $\lambda \leq \boldsymbol{\delta}^2_1$. Following Becker (see [3]), define a measure on $\mathcal{P}_\kappa(\lambda)$ as follows. If $A \subseteq \mathcal{P}_\kappa(\lambda)$, define $\mu(A) = 1$ iff II wins the game where I, II play reals $x_0, x_1, \dots$ coding sets $B_0 \subseteq B_1 \subseteq \dots \in \mathcal{P}_\kappa(\lambda)$ (say, via the coding lemma), and II wins the run of the game provided $B = \bigcup_{n\in\omega} B_n \in A$. By a result of Becker [2] (see also [3]), this game is determined from AD (or one may assume here $AD^{\mathbb{R}}$). As in [3], a simple dovetailing argument shows that the κ additivity of μ implies the normality of μ. To show κ additivity, let A_α, $\alpha < \bar{\kappa}$, be μ measure one sets. Let $F\colon \bar{\kappa}^\omega \to \bar{\kappa}^\omega \times \omega^\omega$ be Lipschitz continuous such that $\forall \alpha < \bar{\kappa}\ \forall$ honest S containing α and closed under F $\forall s$ enumerating S $[F(\alpha^\frown s)$ codes a winning strategy for II for the measure one set $A_\alpha]$.

To see I cannot win the game corresponding to $A = \bigcap_{\alpha<\bar{\kappa}} A_\alpha$, have II play as follows. If I, II have played $x_0, x_1, \dots, x_{2n-2}, x_{2n-1}$, coding $B_0, B_1, \dots, B_{2n-1}$, and I then plays x_{2n} coding B_{2n}, II plays x_{2n+1} coding B_{2n+1} such that $\forall \alpha < \bar{\kappa}$ $\forall^* S\ \forall^* s\ [F(\alpha^\frown s)(x_0, \dots, x_{2n}) \subseteq B_{2n+1}]$. This is possible by the previous remark. Using countable additivity of the supercompactness measure and category, it easy to see that this defeats I's winning strategy for the set A.

We present one more result in this section, which was not needed in the previous arguments but seems to be of related interest.

Theorem 3.2. *Let κ be a Suslin cardinal, $b\colon \kappa \to \mathrm{ON}$ be such that $\forall \alpha < \kappa\ b(\alpha)$ is a cardinal $\geq \omega_1$, and $g\colon \kappa \to \mathrm{ON}$ be a function with $b(\alpha) \leq g(\alpha) < b(\alpha)^+$ for all $\alpha < \kappa$. Then there is a function h with domain κ such that for all $\alpha < \kappa$, $h(\alpha)$ is a well-ordering of $j_\nu(b(\alpha))$ of rank $|h(\alpha)| \geq g(\alpha)$.*

Proof. By playing the usual game, there is a strategy $F\colon \kappa^\omega \to \omega^\omega$ such that $\forall \alpha < \kappa$ $\forall s \in \kappa^\omega\ F(\alpha^\frown s)$ is a well-ordering of $b(\alpha)$, and $\forall \alpha < \kappa\ \forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^* s \in S^\omega$ $|F(\alpha^\frown s)| \geq g(\alpha)$. For $\beta < b(\alpha)$, we let $F(\alpha^\frown s)(\beta)$ denote the rank of β in the well-ordering $F(\alpha^\frown s)$.

We define, uniformly in $\alpha < \kappa$, a tree T_α. We define $((p_0, \dots, p_n), (\beta_0, \dots, \beta_n)) \in T_\alpha$ iff:

(1) $p_0, \dots, p_n \in \kappa^{<\omega}$, and p_{i+1} extends p_i for all $0 \leq i < n$.

(2) $\beta_0, \dots, \beta_n < j_\nu(b(\alpha))$, and for all $0 \leq i < n$ we have $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^*_{p_{i+1}} s \in S^\omega\ (\beta_{i+1}(S), \beta_i(S)) \in F(\alpha^\frown s)$.

(3) $\forall 0 \leq i \leq n\ \exists \gamma_i\ \forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^*_{p_i} s \in S^\omega\quad F(\alpha^\frown s)(\beta_i(S)) = \gamma_i(S)$.

It is easy to see that each T_α is well-founded, since if $(p_0, \beta_0; p_1, \beta_1; \dots)$ were a branch through T_α, and $\gamma_0, \gamma_1, \dots$ are as in (3), then $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ [\gamma_0(S) > \gamma_1(S) > \dots]$, a contradiction.

To finish the proof, fix $\alpha < \kappa$, and we show that the rank of T_α is at least $g(\alpha)$. Let $\prec$ be a well-ordering of $b(\alpha)$ of length $g(\alpha)$. Let $T_\prec$ be the tree of finite decreasing sequences $(\delta_0, \dots, \delta_n)$ with respect to $\prec$. We define an embedding π from $T_\prec$ into T_α. Let $\pi(\emptyset) = \emptyset$. Assume $\pi(\delta_0, \dots, \delta_n) = ((p_0, \dots, p_n), (\beta_0, \dots, \beta_n))$ has been defined, and assume inductively that $\forall^* S\ \forall^*_{p_n} s \in S^\omega\ F(\alpha^\frown s)(\beta_n(S)) \geq |(\delta_0, \dots, \delta_n)|_{T_\prec}$. If $\delta_{n+1} \prec \delta_n$, then by additivity of category and normality of ν it follows that $\exists p_{n+1}$ extending p_n $\exists \beta_{n+1} < j_\nu(b(\alpha))$ $\exists \gamma_{n+1} \in \mathrm{ON}$ such that

$$\forall^* S\ \forall^*_{p_{n+1}} s \in S^\omega\ [(\beta_{n+1}(S), \beta_n(S)) \in F(\alpha^\frown s) \wedge \\ F(\alpha^\frown s)(\beta_{n+1}(S)) = \gamma_{n+1}(S) \geq |(\delta_0, \dots, \delta_{n+1})|_{T_\prec}].$$

Set $\pi(\delta_0, \dots, \delta_{n+1}) = ((p_0, \dots, p_n, p_{n+1}), (\beta_0, \dots, \beta_n, \beta_{n+1}))$ for the least such p_{n+1}, β_{n+1}, γ_{n+1}. □

Corollary 3.3. *Let κ be a Suslin cardinal, $b \colon \kappa \to \mathrm{ON}$ such that $\forall \alpha < \kappa\ b(\alpha)$ is a limit Suslin cardinal with $\mathrm{cof}(b(\alpha)) > \kappa$, and $g \colon \kappa \to \mathrm{ON}$ be a function with $b(\alpha) \leq g(\alpha) < b(\alpha)^+$ for all $\alpha < \kappa$. Then there is a function h with domain κ such that for all $\alpha < \kappa$, $h(\alpha)$ is a well-ordering of $b(\alpha)$ of rank $|h(\alpha)| \geq g(\alpha)$.*

Proof. This follows immediately since if ρ is a limit Suslin cardinal of cofinality $> \kappa$, then $j_\nu(\rho) = \rho$. We use here the fact that $j_\nu(\rho) = \sup_{\rho' < \rho} j_\nu(\rho')$, which follows from lemma 1.3 since $\mathrm{cof}(\rho) > \kappa$. Also, for any Suslin cardinal $\rho' > \kappa$, $j_\nu(\rho')$ is less than the second Suslin cardinal greater than ρ', which follows from a direct computation of the ultrapower (for example, by coding functions $f \colon \mathcal{P}_{\omega_1}(\kappa) \to \rho'$ by coding Lipschitz continuous $F \colon \kappa^\omega \to \omega^\omega$ such that $\forall^* S\ \forall^* s \in S^\omega\ |F(s)| = f(S)$). □

We close with some observations on the preceding result. First, if $\kappa = \omega_1$, then we may take, in defining T_α, $\beta_i < b(\alpha)$. That is, the integration with respect to the supercompactness measure ν is not necessary. To see this, for each $\alpha < \omega_1$, let $\alpha' \geq \alpha$ be the least ordinal which is honest as a subset of ω_1. Then, in the above proof, replace expressions $\forall^* S\ \forall^*_{p_i} s \in S^\omega\ (\dots)$ by $\forall^*_{p_i} s \in (\alpha')^\omega\ (\dots)$. Thus we have:

Corollary 3.4. *Let $b \colon \omega_1 \to \mathrm{ON}$ be such that $\forall \alpha < \omega_1\ b(\alpha)$ is a cardinal $\geq \omega_1$, and $g \colon \omega_1 \to \mathrm{ON}$ be a function with $b(\alpha) \leq g(\alpha) < b(\alpha)^+$ for all $\alpha < \omega_1$. Then there is a function h with domain ω_1 such that for all $\alpha < \omega_1$, $h(\alpha)$ is a well-ordering of $b(\alpha)$ of rank $|h(\alpha)| \geq g(\alpha)$.*

Similarly, if $\mathrm{cof}(\kappa) > \omega$, then in defining T_α we need consider only $S \in \mathcal{P}_{\omega_1}(\alpha')$, where again $\alpha' \geq \alpha$ is the least honest ordinal. It follows in this case that we may relax the requirement in corollary 3.3 to require only $\mathrm{cof}(b(\alpha)) \geq \kappa$.

4. Further Applications

We prove in this section some results on the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\kappa)$, for κ a Suslin cardinal. Also, we consider the relationship between $L(\mathbb{R})$, the ultrapower $\mathcal{M}$ of $L(\mathbb{R})$ by ν, and the generic model $L(\mathbb{R}_G)$. Throughout this section, κ again denotes a fixed Suslin cardinal.

We briefly review the extra facts from [7] that we need in this section. For κ a fixed Suslin cardinal, let G be $L(\mathbb{R})$-generic for the collapse of κ to ω with finite conditions $p \in \kappa^{<\omega}$. G induces an ultrafilter $\mathcal{U}$ on $(\mathcal{P}(\kappa^\omega)/\mathcal{I})^{L(\mathbb{R})}$, where $\mathcal{I}$ is the ideal of weakly meager sets in $\mathcal{P}(\kappa^\omega)$. Recall $A \in \mathcal{P}(\kappa^\omega)$ is weakly meager iff

$\kappa^\omega - A$ is weakly comeager, and B is weakly comeager iff $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^* s \in S^\omega$ $s \in B$. Then, in $V[G]$ we form the ultrapower of $L(\mathbb{R})$, using functions in $L(\mathbb{R})$, via this generic ultrafilter. By [7], this ultrapower is well-founded, and satisfies Los' theorem. Furthermore, after taking the transitive collapse, this ultrapower is equal to $L(\mathbb{R}_G)$, where $\mathbb{R}_G = \omega^\omega \cap V[G]$. Let $\mathcal{M}$ be the transitive collapse of the ultrapower of $L(\mathbb{R})$ by ν. Since $\nu \in L(\mathbb{R})$, $\mathcal{M} \subseteq L(\mathbb{R})$. We define a map $i : \mathcal{M} \to L(\mathbb{R}_G)$ by: $i([F]_\nu) = [F']_\mathcal{U}$, where $F'(s) = F(S)$, $S =$ the range of s. The arguments of [7] show that $i \restriction \mathrm{ON}$ is the identity. [Suppose $p \in \kappa^{<\omega}$ and $H' \colon \kappa^\omega \to \mathrm{ON}$. By additivity of category, for any $S \in \mathcal{P}_{\omega_1}(\kappa)$ there is a $q \in S^{<\omega}$ extending p and an ordinal $H(S)$ such that $\forall^*_q s \in S^\omega\ H'(s) = H(S)$. By normality, there is a $q \in \kappa^{<\omega}$ such that $\forall^*_\nu S\ \forall^*_q s \in S^\omega\ H'(s) = H(S)$. Applying genericity shows that for any $H' \colon \kappa^\omega \to \mathrm{ON}$ in $L(\mathbb{R})$, there is an $H \colon \mathcal{P}_{\omega_1}(\kappa) \to \mathrm{ON}$ in $L(\mathbb{R})$ such that $\forall^*_\mathcal{U} s \in \kappa^\omega\ H'(s) = H(S)$, where $S = \mathrm{range}(s)$.]

Likewise, $i \restriction \mathcal{P}(\alpha)$ is the identity for all $\alpha \in \mathrm{ON}$. In particular, any set of ordinals in $\mathcal{M}$ is also in $L(\mathbb{R}_G)$.

Remark 4.1. Although $\mathbb{R} \subseteq \mathbb{R}_G$, $L(\mathbb{R}) \subsetneq L(\mathbb{R}_G)$, and in fact the inclusion fails even for sets of ordinals. For example, if $\kappa = \boldsymbol{\delta}^1_{2n+1}$ ($n \geq 1$), then arguments of [7] (see also lemma 4.3 of [3]) show that $\boldsymbol{\delta}^1_{2n+2} < j_\nu(\kappa) < (\boldsymbol{\delta}^1_{2n+2})^+$. By the above comments, $\lambda \doteq j_G(\kappa) = j_\nu(\kappa)$, where j_G is the embedding from the generic ultrapower. Since j_G is elementary, $L(\mathbb{R}_G) \models \lambda$ is a cardinal. Thus, any well-order of $\boldsymbol{\delta}^1_{2n+2}$ of length λ in $L(\mathbb{R})$ cannot lie in $L(\mathbb{R}_G)$.

The next result is an immediate corollary of the proof of theorem 2.1 (that proof, minus the last paragraph, did not use the fact that that $\mathrm{cof}(f(\alpha)) \leq \kappa$ and showed that $C = \mathrm{range}(f) \in \mathcal{M}$).

Theorem 4.1. *Suppose $A \subseteq \lambda < \Theta$, $|A| \leq \kappa$. Then $A \in \mathcal{M}$. In particular, $A \in L(\mathbb{R}_G)$ as well.*

We note that theorem 4.1 implies $\boxminus_{\kappa,\lambda}$ for all $\lambda < \Theta$ as well. To see this, let $f \colon \kappa \to \lambda$ be such that $\forall \alpha < \kappa\ \mathrm{cof}(f(\alpha)) \leq \kappa$. Then by 4.1, $f \in \mathcal{M}$, and so $f \in L(\mathbb{R}_G)$. Since $L(\mathbb{R}_G)$ satisfies countable choice (by elementarity, since $L(\mathbb{R})$ does), and κ is countable in $L(\mathbb{R}_G)$, there is an $A \in L(\mathbb{R}_G)$, A countable in $L(\mathbb{R}_G)$, such that $\forall \alpha < \kappa\ A \cap f(\alpha)$ is cofinal in $f(\alpha)$. Since the forcing has size κ, there is a set $B \in L(\mathbb{R})$, $|B| \leq \kappa$, with $A \subseteq B$.

The next lemma says that $\mathcal{M}$ and $L(\mathbb{R}_G)$ compute the same successors for cardinals of the form $j_\nu(\lambda)$ for λ uncountable (since $j_\nu(\lambda)^{+L(\mathbb{R}_G)} = j_G(\lambda)^{+L(\mathbb{R}_G)} = j_G(\lambda^+) = j_\nu(\lambda^+)$, the second equality by elementarity).

Lemma 4.1. *For all cardinals $\lambda > \omega$, $j_\nu(\lambda^+) = (j_\nu(\lambda)^+)^\mathcal{M}$.*

Proof. Let $\eta < j_\nu(\lambda^+)$, say $\eta = [F]_\nu$ where $F(S) < \lambda^+$ for all $S \in \mathcal{P}_{\omega_1}(\kappa)$. By the usual game argument, there is a Lipschitz continuous $\tau \colon \kappa^\omega \to \omega^\omega$ such that for all $s \in \kappa^\omega$ $\tau(s)$ codes a well-ordering $\prec_{\tau(s)}$ of λ, and $\forall^* S$ and any enumeration s of S, $|\prec_{\tau(s)}| > F(S)$. We define a function V such that for all $S \in \mathcal{P}_{\omega_1}(\kappa)$ $V(S)$ is a well-ordering of λ, and $\forall^* S\ |V(S)| > F(S)$, which suffices. Actually, $V(S)$ will be a well-founded tree on $S^{<\omega} \times \lambda$ (which we can uniformly identify with λ, as λ is uncountable). Define $((p_0, \dots, p_n), (\beta_0, \dots, \beta_n)) \in V(S)$ iff $\forall i < n$ we have p_{i+1} extends p_i, $\forall^*_{p_{i+1}} s \in S^\omega\ \beta_{i+1} \prec_{\tau(s)} \beta_i$, and $\forall i \leq n\ \exists \gamma_i\ (\forall^*_{p_i} s \in S^\omega\ |\beta_i|_{\prec_{\tau(s)}} = \gamma_i)$. Easily $V(S)$ is well-founded for all $S \in \mathcal{P}_{\omega_1}(\kappa)$. Also, for S such that for all

s enumerating S, $| \prec_{\tau(s)} | > F(S)$, it follows as in theorem 3.2 that $|V(S)| > F(S)$. $\square$

Remark 4.2. Let $\kappa = \boldsymbol{\delta}^1_{2n+1}$. Using the proof of theorem 3.2, one can see that $j_\nu(\omega_1) = \boldsymbol{\delta}^1_{2n+2}$. Note that $\boldsymbol{\delta}^1_{2n+2}$ is a cardinal in both $L(\mathbb{R})$ and $L(\mathbb{R}_G)$ (the latter since the forcing is κ^+-c.c.). Remark 4.1 shows that these models do not compute the same successor of $\boldsymbol{\delta}^1_{2n+2}$, though by lemma 4.1 $\mathcal{M}$ and $L(\mathbb{R}_G)$ do.

In this paper we have made use of the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\kappa)$ to obtain results (such as $\boxminus_{\kappa,\lambda}$ for $\lambda < \Theta$) not mentioning the supercompactness measure. We can also use the results here to obtain results about ν. The next theorem illustrates this. It is a covering result for the supercompactness measure ν.

Theorem 4.2. *Let κ be a Suslin cardinal, $\lambda < \Theta$. Then for any $F\colon \mathcal{P}_{\omega_1}(\kappa) \to \lambda$, there is an $A \subseteq \lambda$, $|A| \leq \kappa$, such that $\forall^*_\nu S \in \mathcal{P}_{\omega_1}(\kappa)\ F(S) \in A$.*

Corollary 4.1. *For any $F\colon \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1}) \to \lambda < \Theta$, there is an A of size $\leq \boldsymbol{\delta}^1_{2n+1}$ such that $\forall^*_\nu S \in \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^1_{2n+1})\ F(S) \in A$.*

Proof of theorem 4.2. Fix F as in the theorem. Let $F'\colon \kappa^\omega \to \kappa^\omega \times \omega^\omega$ be Lipschitz continuous such that if T is honest, closed under F', $\alpha \in T$, and t enumerates T, then $u = F'(\alpha^\frown t)$ codes $B_{F'(\alpha^\frown t)} = \{F(T)\}$. Applying remark 3.2 to F' gives the result. $\square$

The next theorem relates the models $\mathcal{M}$ (the transitive collapse of the ultrapower of $L(\mathbb{R})$ by the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\kappa)$), the ground model $L(\mathbb{R})$, and the generic ultrapower $L(\mathbb{R}_G)$ using the collapse of κ to ω. This theorem does not use our previous results, but is of related interest.

Theorem 4.3. *Let κ be a Suslin cardinal. Then for all $\lambda \in \mathrm{ON}$, $\mathcal{M} \cap \mathcal{P}(\lambda) = L(\mathbb{R}) \cap L(\mathbb{R}_G) \cap \mathcal{P}(\lambda)$.*

Proof. $\mathcal{M} \subseteq L(\mathbb{R})$, and by our previous remarks, any set of ordinals in $\mathcal{M}$ lies in $L(\mathbb{R}_G)$, hence $\mathcal{M} \cap \mathcal{P}(\lambda) \subseteq L(\mathbb{R}) \cap L(\mathbb{R}_G) \cap \mathcal{P}(\lambda)$. Fix now $A \subseteq \lambda$, $A \in L(\mathbb{R}) \cap L(\mathbb{R}_G)$. Fix a function $F\colon \kappa^\omega \to \mathrm{ON}$ representing A in the generic ultrapower (so $F \in L(\mathbb{R})$). We construct a function $H\colon \mathcal{P}_{\omega_1}(\kappa) \to \mathrm{ON}$ such that $[H]_\nu = A$. For $p \in \kappa^{<\omega}$, $\alpha < \lambda$, we say p is α-good if either $\alpha \in A$ and $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \neg\forall^*_p s \in S^\omega\ \alpha(S) \notin F(s)$ or $\alpha \notin A$ and $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)\ \neg\forall^*_p s \in S^\omega\ \alpha(S) \in F(s)$. We say p is good if it is α-good for all $\alpha < \lambda$. Note that good p's exist, since any p in the generic G (viewed as a function from ω onto κ) is good. Define then $H(S)$ by: $\beta \in H(S) \leftrightarrow \exists p \in S^{<\omega}$ [p is good $\wedge \forall^*_p s \in S^\omega\ \beta \in F(s)$]. We claim that $A = [H]_\nu$. Suppose $\alpha \in A$. By genericity, there is either a $p \in G$ such that $\forall^* S\ \forall^*_p s \in S^\omega\ \alpha(S) \in F(s)$, or there is a $p \in G$ such that $\forall^* S\ \forall^*_p s \in S^\omega\ \alpha(S) \notin F(s)$. The latter case, however, is impossible, since the function $s \to \alpha(S)$, where S is the range of s, represents α in the generic ultrapower, and F represents A. Since $p \in G$ is good, $\alpha \in [H]_\nu$. Thus, $A \subseteq [H]_\nu$. Conversely, suppose $\alpha \in [H]_\nu$. By normality, there is a good $p \in \kappa^{<\omega}$ such that $\forall^* S\ \forall^*_p s \in S^\omega\ \alpha(S) \in F(s)$. Since p is α-good, we must have $\alpha \in A$. $\square$

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