The Strong Partition Relation on ω_1 Revisited

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We give a new proof of the strong partition relation on ω_1, assuming the axiom of determinacy, which uses only a general argument not involving the complete analysis of a measure on ω_1.

We assume the axiom of determinacy (AD) throughout, as well as dependent choice (DC). Martin showed that under these hypotheses, ω_1 has the strong partition relation, $\omega_1 \to (\omega_1)^{\omega_1}$, defined precisely below. Martin's proof used the theory of indiscernibles for L to code functions from ω_1 to ω_1, and also developed a general method for proving partition relations from AD. Since then several different proofs of the strong partition relation on ω_1 have been found, although all rely on Martin's general method. Kechris [4] gave a proof using generic codes for countable ordinals, and the first author [1] gave a proof using an analysis of measures on ω_1 to develop the coding of functions. The latter proof is the only one known to generalize to the higher odd projective ordinals, the $\boldsymbol{\delta}^1_{2n+1}$, and a ways beyond. These generalizations require a detailed inductive analysis (see [3]), which currently extends only a relatively small way into the $L(\mathbb{R})$ hierarchy of pointclasses. This has the unfortunate consequence that past this point all is lost. In particular, it is not known if there is a projective-like hierarchy past the ordinal of the inductive sets for which all of the analogs of the odd projective ordinals have the strong partition relation.

We give here yet another proof of the strong partition relation on ω_1. This proof is in some sense a derivative of the proof in [1], although it differs from that proof in a potentially significant way. It develops a coding of the functions on ω_1 using not the full analysis of measures on ω_1 (as the proof in [1]), but rather only the first step in that analysis, which is a general step not involving the detailed combinatorics at the cardinal. Thus, this proof uses only the first trivial step in the analysis of the measure, together with a "cheap" coding for the rest of the analysis. The idea is to find a proof which does not depend so much on the complete inductive analysis, and thus might be applicable to cardinals past the point to which the current inductive theory applies. In other words, the goal is to find a method for proving the strong partition relation at a cardinal κ which "decouples" the proof from the detailed inductive analysis below κ. This would have many interesting consequences.

Although this seems promising, we do not at the moment know how to extend this proof to even $\boldsymbol{\delta}^1_3$, at least not without invoking the detailed inductive analysis we are trying to avoid. We are not sure, therefore, if the goal mentioned above is attainable, or even reasonable, but we feel the current proof suggests an interesting possibility.

We employ fairly standard terminology throughout. In particular, ω^ω denotes the Baire space of functions from ω to ω, which serve as our "reals." ON denotes the class of ordinals. A pointclass $\boldsymbol{\Gamma}$ is

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a collection of subsets of ω^ω closed under continuous preimage, and $\check{\boldsymbol{\Gamma}} = \{\omega^\omega - A \colon A \in \boldsymbol{\Gamma}\}$ denotes the dual pointclass. We say $\boldsymbol{\Gamma}$ is non-selfdual if $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$. A tree on a set X means a set $T \subseteq X^{<\omega}$ closed under subsequence. We view members of a tree on $X \times Y$ as also being pairs of finite sequences from X and Y of the same length. We let $z \to (z_0, z_1, \dots)$ denote a recursive bijection between ω^ω and $(\omega^\omega)^\omega$.

1 Review of Martin's Theorem

We first review the strong partition property and state a form of Martin's theorem which gives abstract hypotheses for a coding of functions on an infinite cardinal κ which guarantees κ has the strong partition property.

We say a function $f \colon \kappa \to \mathrm{ON}$ has *uniform cofinality* ω if there is a function $f' \colon \kappa \to \mathrm{ON}$ such that $f'(\beta) < f'(\beta+1)$ for all $\beta < \kappa$ and $f(\alpha) = \sup_{\beta < \omega\cdot(\alpha+1)} f'(\beta)$ for all $\alpha < \kappa$. We say $f \colon \kappa \to \mathrm{ON}$ is of the *correct type* if f is strictly increasing, discontinuous ($f(\alpha) > \sup_{\beta<\alpha} f(\beta)$ for α limit), and f has uniform cofinality ω.

We take as our definition of the *strong partition relation* on κ the statement that for every partition P of the functions from κ to κ of the correct type into two pieces there is a c.u.b. $C \subseteq \kappa$ such that P is constant on the functions from κ to C of the correct type. This is easily seen to be equivalent to the more standard definition that for any partition of the increasing functions into two pieces there is a homogeneous set of size κ.

By a *coding* of the functions from κ to κ we mean a map $\phi \colon \omega^\omega \to \wp(\kappa \times \kappa)$ such that every function $f \colon \kappa \to \kappa$ is in the range of ϕ. Given a coding map ϕ, we define for each $\alpha, \beta < \kappa$ two associated sets R_α and $R_{\alpha,\beta}$ by

$$
\begin{aligned}
& x \in R_{\alpha,\beta} \iff \phi(x)(\alpha,\beta) \wedge \forall \beta' < \kappa\ (\phi(x)(\alpha,\beta') \to \beta' = \beta) \\
\text{and} \qquad & x \in R_\alpha \iff \exists \beta < \kappa\ (x \in R_{\alpha,\beta}).
\end{aligned}
$$

If $x \in R_\alpha$, let $\phi(x)(\alpha)$ be the unique β such that $\phi(x)(\alpha,\beta)$. Define $x \in S_{\alpha,\beta} \iff \forall \alpha' \leq \alpha\ \exists \beta' \leq \beta\ (x \in R_{\alpha',\beta'})$, and $x \in S_\alpha \iff \forall \alpha' \leq \alpha\ (x \in R_{\alpha'})$.

Definition 1.1 A regular cardinal κ is *reasonable* if there is a nonselfdual pointclass $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$ and a coding map $\phi \colon \omega^\omega \to \wp(\kappa \times \kappa)$ satisfying:

1. $\forall \alpha, \beta < \kappa\ R_{\alpha,\beta} \in \boldsymbol{\Delta} \doteq \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$.
2. If $\alpha < \kappa$, $A \in \exists^{\omega^\omega} \boldsymbol{\Delta}$, and $A \subseteq R_\alpha$, then $\exists \beta' < \kappa\ \forall x \in A\ \exists \beta < \beta'\ R_{\alpha,\beta}(x)$.

Theorem 1.2 (Martin) *Every reasonable cardinal has the strong partition relation.*

Proof. We sketch the proof assuming some additional closure properties of $\boldsymbol{\Gamma}$ which are trivially satisfied in the case where $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}^1_1$ and $\kappa = \omega_1$ which we will eventually use (the more general case can be found in [3]).

Let $\boldsymbol{\Gamma}$ and $\phi \colon \omega^\omega \to \wp(\kappa \times \kappa)$ witness that κ is reasonable. Suppose P is a partition of functions from κ to κ of the correct type. Consider the integer game where I plays out $x \in \omega^\omega$, II plays out $y \in \omega^\omega$, and if there exists a least $\alpha < \kappa$ such that either $x \notin S_\alpha$ or $y \notin S_\alpha$, then II wins exactly when $x \notin S_\alpha$. Otherwise, both $\phi(x)$ and $\phi(y)$ determine functions from κ to κ, and we can define for $\alpha < \kappa$

$$
F_{x,y}(\alpha) = \sup_{\beta < \omega\cdot(\alpha+1)} \max\bigl(\phi(x)(\beta), \phi(y)(\beta)\bigr).
$$

If $F_{x,y}$ is not of the correct type, then II wins. Otherwise, II wins iff $P(F_{x,y}) = 1$.

Assume II has a winning strategy τ (the argument for I is slightly simpler). Note that if $x \in S_{\alpha,\beta}$, then $\tau(x) \in S_\alpha$. Define $\ell(\alpha,\beta) = \sup\{\phi(y)(\alpha) : y \in \tau[S_{\alpha,\beta}]\}$. $\boldsymbol{\Delta}$ is closed under $< \kappa$ unions, which we assume here. So $S_{\alpha,\beta} \in \boldsymbol{\Delta}$, and hence $\tau[S_{\alpha,\beta}] \in \exists^{\omega^\omega} \boldsymbol{\Delta}$. Since κ is reasonable, $\ell(\alpha,\beta) < \kappa$. Let $C \subseteq \kappa$ be a c.u.b. set closed under ℓ, and let C' be the limit points of C. We claim that for any $F' \colon \kappa \to C'$ of the correct type, $P(F') = 1$. So suppose $F' \colon \kappa \to C'$ is of the correct type. Fix $F \colon \kappa \to C$ such that for all $\alpha' < \kappa$, $F'(\alpha') = \sup_{\alpha < \omega \cdot (\alpha'+1)} F(\alpha)$, and fix $x \in \omega^\omega$ such that $F = \phi(x)$. Let $y = \tau(x)$. So, $\phi(y)$ is a function from κ to κ, and $\phi(y)(\alpha) \le \phi(x)(\alpha+1)$ for all $\alpha < \kappa$. Thus, $F' = F_{x,y}$, and since τ was a winning strategy for II, $P(F') = 1$. $\square$

2 Coding Subsets of ω_1

We describe a method to code an arbitrary subset of ω_1 by a real in ω^ω. We first recall some facts and fix some terminology. For a tree T on $\omega \times \alpha$, the projection of the set of paths through the tree $p[T]$ is $\{x \in \omega^\omega \colon \exists f \in \alpha^\omega\ \forall n\ (x \restriction n, f \restriction n) \in T\}$. For $x \in \omega^\omega$, the section of the tree T_x is $\{s \in \alpha^{<\omega} : (x \restriction \mathrm{lh}(s), s) \in T\}$. For $\beta \le \alpha$, the restriction $T \restriction \beta$ means $T \cap (\omega \times \beta)^{<\omega}$. We generalize this notation in an obvious way to trees on other products of ordinals. If a node s is in the wellfounded part of T (written $\mathrm{wf}(T)$), the rank of s in T is denoted $|T(s)|$.

Recall $\mathrm{WO} \subseteq 2^\omega$ is the standard set of codes for countable ordinals. For $w \in \mathrm{WO}$, let $|w| < \omega_1$ be the ordinal it codes, and for $n \in \omega$, let $|n|_w$ denotes the rank of n in the wellorder coded by w. Let $\mathrm{WO}_{\le\alpha} = \{w \in \mathrm{WO} \colon |w| \le \alpha\}$ and likewise for $\mathrm{WO}_{<\alpha}$. Let T^{WO} on $\omega \times \omega_1$ be a *Shoenfield tree* of WO, i.e. $p[T^{\mathrm{WO}}] = \mathrm{WO}$ and $\forall \alpha < \omega_1 \exists x \in \omega^\omega\ |x| = \alpha \wedge T^{\mathrm{WO}}_x \restriction \alpha$ is illfounded. Recall this tree is constructed as the tree of initial segments of (x, f) where x codes a linear order of ω, and $f \colon \omega \to \omega_1$ is order preserving with respect to this linear ordering.

A standard fact from determinacy is the existence of a *Kunen tree*, T^K. This is a tree on $\omega \times \omega_1$ such that for any $f \colon \omega_1 \to \omega_1$ there is an $x \in \omega^\omega$ such that T^K_x is wellfounded and $\forall \alpha \ge \omega\ |T^K_x \restriction \alpha| \ge f(\alpha)$. The reader may consult [5] or [3] for the details of the construction. To avoid minor but annoying notational complications arising from considering ordinals versus finite sequences of ordinals, we find it convenient to employ a linear-order version of the Kunen tree. This is obtained by considering the Kleene-Brouwer order on T^K_x and taking a bijection between ω_1 and $\omega_1^{<\omega}$. We also make this a total order on ω_1 by adding all ordinals coding sequences not in T^K_x, in their natural ordering as ordinals, to the beginning of the ordering. We will continue to let T^K_x denote this linear ordering of ω_1, and let $|T^K_x \restriction \alpha|$ denote the rank of T^K_x restricted to α (provided this is wellfounded).

More precisely, T^K is a function which assigns to each $x \in \omega^\omega$ a linear ordering T^K_x of ω_1 and which satisfies:

1) For every $f \colon \omega_1 \to \omega_1$ there is an $x \in \omega^\omega$ such that T^K_x is a wellordering and for all $\alpha \ge \omega$, $|T^K_x \restriction \alpha| > f(\alpha)$.

2) For all $\alpha, \beta < \omega_1$, $\{x \colon (\alpha, \beta) \in T^K_x\}$ is clopen.

From (2) we recall a standard computation shows that for any $\alpha, \beta < \omega_1$, the set of x such that $T^K_x \restriction \alpha$ is wellfounded of rank $< \beta$ is $\mathbf{\Delta}^1_1$.

Recall that there is a unique normal measure on ω_1, which we denote W^1_1. This follows from the weak partition relation on ω_1, which we assume here, and is an easy consequence of AD. These basic facts are proved, for example, in [3].

Define WO^* to be $\{z \in \omega^\omega : z_0 \in \mathrm{WO} \wedge z_1 \in 2^\omega\}$. Each $z \in \mathrm{WO}^*$ codes a bounded subset $A_z \subseteq |z_0|$ by

$$\beta \in A_z \iff \exists n\ (|n|_{z_0} = \beta \wedge z_1(n) = 1).$$

We say $z, z' \in \omega^\omega$ are *compatible codes* iff $z, z' \in \mathrm{WO}^*$ and for all $m, m' < \omega$, if $|m|_{z_0} = |m'|_{z'_0}$, then $z_1(m) = z'_1(m')$. So, the predicate S representing incompatibility of codes is

$$S(z, z') \iff \big(z \notin \mathrm{WO}^* \vee z' \notin \mathrm{WO}^* \vee \exists m, m' < \omega\, [|m|_{z_0} = |m'|_{z'_0} \wedge z_1(m) \ne z'_1(m')]\big).$$

Note that S is $\mathbf{\Sigma}^1_1$, so there is a tree T^S on ω^3 with $p[T^S] = S$.

View every real τ as coding a strategy for II in an integer game. We write $\tau(x)$ for II's response against I's play of x. We say τ is a $\le \alpha$ *invariant strategy* if $\forall x, x' \in \mathrm{WO}_{\le\alpha}\ \neg S(\tau(x), \tau(x'))$. If in addition τ satisfies $\forall x \in \mathrm{WO}_{\le\alpha}\ |\tau(x)_0| \ge |x|$, then τ is a *good* $\le \alpha$ *invariant strategy.* Furthermore, τ is a *good invariant strategy* if τ is a good $\le \alpha$ invariant strategy for all $\alpha < \omega_1$.

We now give our "cheap" coding of the subsets of ω_1. This will be combined momentarily with the first step in the analysis of a measure to yield our final coding of the subsets of ω_1. Say τ *codes* $B \subseteq \omega_1$ if

$$\forall x \in \mathrm{WO}\ \big(\tau(x) \in \mathrm{WO}^* \wedge |\tau(x)_0| \geq |x| \wedge \\ \forall \beta < |\tau(x)_0|\ \forall m < \omega\ \ [\beta = |m|_{\tau(x)_0} \to (\beta \in B \leftrightarrow \tau(x)_1(m) = 1)]\big).$$

Clearly, if τ codes $B \subseteq \omega_1$, then τ is good and invariant. Conversely, if τ is good and invariant, then there exists a unique $B \subseteq \omega_1$ which is coded by τ.

Lemma 2.1 *For all $B \subseteq \omega_1$ there exists a good invariant strategy τ which codes B.*

Proof. Play the Solovay game where I plays x, II plays y, and II wins iff

$$x \in \mathrm{WO} \to \big([y \in \mathrm{WO}^* \wedge |y_0| \geq |x| \wedge \\ \forall \beta < |y_0|\ \forall m < \omega\ [\beta = |m|_{y_0} \to (\beta \in B \leftrightarrow y_1(m) = 1)]\big).$$

By boundedness I cannot have a winning strategy. A winning strategy τ for II codes B. □

The main idea for connecting the cheap coding with the first step in the analysis of a measure is embodied in the definition of the following tree T^I, which attempts to witness a failure of good invariance. Define $S'(y, y', z, z') \iff S(z, z') \vee L(z_0, y) \vee L(z'_0, y')$, where L is a $\mathbf{\Sigma}^1_1$ relation such that if $x, y \in \mathrm{WO}$, then $L(x, y) \iff |x| < |y|$. $S' \in \mathbf{\Sigma}^1_1$, and we let $T^{S'}$ be a tree on ω^5 projecting to S'. Define T^I on $\omega \times (\omega \times \omega_1)^2 \times \omega^3$ by

$$(t, a, \alpha, a', \alpha', b, b', c) \in T^I \iff \\ (a, \alpha) \in T^{\mathrm{WO}} \wedge (a', \alpha') \in T^{\mathrm{WO}} \wedge ta = b \wedge ta' = b' \wedge (a, a', b, b', c) \in T^{S'},$$

where $ta = b$ means that in some reasonable manner t codes a partial strategy and b is consistent with t applied to a.

Lemma 2.2 *For all $\tau \in \omega^\omega$ and for all $\beta < \omega_1$, $T^I_\tau \restriction \beta$ is wellfounded iff τ is a good $\leq \beta$ invariant strategy.*

Proof. If $T^I_\tau \restriction \beta$ is illfounded, let $(x, \alpha, x', \alpha', z, z', w) \in [T^I_\tau \restriction \beta]$. So $(x, \alpha), (x', \alpha') \in [T^{\mathrm{WO}}]$ and $x, x' \in \mathrm{WO}_{\leq\beta}$. Also, $\tau(x) = z$ and $\tau(x') = z'$, where $(x, x', z, z', w) \in T^{S'}$. If either z or z' is not in WO^*, then τ is not $\leq \beta$ invariant. Otherwise, either $|z_0| < |x|$ or $|z'_0| < |x'|$ or $S(z, z')$ holds. Thus, τ cannot be good $\leq \beta$ invariant.

Conversely, suppose τ is not good $\leq \beta$ invariant. Then there exist $x, x' \in \mathrm{WO}$ with $|x|, |x'| \leq \beta$ and either $\tau(x) \in \mathrm{WO}$ and $|\tau(x)_0| < |x|$, or similarly for x', or else $S(\tau(x), \tau(x'))$. Thus, $S'(x, x', \tau(x), \tau(x'))$, and hence there exists $w \in \omega^\omega$ such that $(x, x', \tau(x), \tau(x'), w) \in [T^{S'}]$. Choose ordinal sequences α, α' such that for all n, $\alpha_n, \alpha'_n < \beta$ and $(x, \alpha), (x', \alpha') \in [T^{\mathrm{WO}}]$. Then $(x, \alpha, x', \alpha', \tau(x), \tau(x'), w) \in [T^I_\tau \restriction \beta]$, showing $T^I_\tau \restriction \beta$ is illfounded. □

The last ingredient we need for our cheap coding is a coding of the c.u.b. subsets of ω_1. This can be done in several ways. Suppose $\sigma \in \omega^\omega$ and T^K_σ is wellfounded. Let C_σ be the c.u.b. subset of ω_1 of limit ordinals closed under $\alpha \mapsto |T^K_\sigma \restriction \alpha|$. The next lemma shows that such C_σ's form a base of the c.u.b. subsets of ω_1.

Lemma 2.3 *If $C \subseteq \omega_1$ is c.u.b., then there exists $\sigma \in \omega^\omega$ such that T^K_σ is wellfounded and $C_\sigma \subseteq C$.*

Proof. Apply property (1) of the Kunen tree to the function that maps α to the least element in C greater than α. □

The notion of a simple set goes back to an idea of Kunen used in the proof of a $\mathbf{\Delta}^1_3$ coding of ω_ω. It is also used in the proof of [1]. Given three reals $\tau, \sigma, x \in \omega^\omega$, we define a corresponding *simple set* $S_{\tau,\sigma,x} \subseteq \omega_1$ by:

$$\begin{aligned}\alpha \in S_{\tau,\sigma,x} \iff & \exists \text{ limit } \gamma \leq \alpha \\ & \forall\delta < \gamma(|T^K_\sigma \upharpoonright \delta| < \gamma \wedge |T^I_\tau \upharpoonright \delta| < \gamma) \wedge \\ & \exists\delta < \gamma\ (\alpha = |T^K_x \upharpoonright \gamma(\delta)| \wedge \exists y \in \mathrm{WO}_{<\gamma} \exists n < \omega\ |n|_{\tau(y)_0} = \delta \wedge \tau(y)_1(n) = 1).\end{aligned}$$

Here $|T(\beta)| = \gamma$ means that β is in the wellfounded part of T and the rank of β in T is γ. Similarly, $|T| < \beta$ means that $|T|$ is wellfounded and its rank is less than β. In this definition we have in mind (although it is not necessarily the case) that τ is a good invariant strategy coding a set $B \subseteq \omega_1$, σ codes a c.u.b. $C \subseteq \omega_1$, and x codes via the Kunen tree a function from ω_1 to ω_1. Intuitively, $S_{\tau,\sigma,x}$ is the image under $T^K_x \upharpoonright \gamma$ of B for various $\gamma \in C$.

We will use the following easy consequence of the normality of W^1_1. A sequence satisfying the conclusion of this lemma is said to *cohere*.

Lemma 2.4 *For each sequence $\langle B_\gamma \subseteq \gamma : \gamma \in C\rangle$ where $C \subseteq \omega_1$ is c.u.b., there is c.u.b. $D \subseteq C$ such that*

$$\forall\gamma_1, \gamma_2 \in D \quad \gamma_1 < \gamma_2 \implies B_{\gamma_1} = B_{\gamma_2} \cap \gamma_1.$$

The next lemma is our key lemma. The idea, roughly, is to make the c.u.b. set carry the burden of witnessing the good invariance of the strategy.

Lemma 2.5 *Every subset of ω_1 is a countable union of simple sets.*

Proof. If not, let $A \subseteq \omega_1$ be a counterexample. Then the collection subsets of A which are contained in a countable union of simple subsets of A form a proper σ ideal on A. By AD we may extend the corresponding filter to a measure. Let ν be a measure on A respecting this ideal. Easily, ν is nonprincipal (every singleton is a simple set). Let $f\colon \omega_1 \to \omega_1$ satisfy the following:

1. There is a ν measure one set $A_1 \subseteq \omega_1$ such that for all $\alpha, \beta \in A_1$ with $\alpha \leq \beta$, $f(\alpha) \leq f(\beta)$.
2. There does not exist a ν measure one set on which f is constant.
3. If $[f']_\nu < [f]_\nu$, then f' does not satisfy (1) and (2).

Since the identity satisfies (1) and (2), there is such a function f, and f is necessarily nonstrictly pressing down. Also note that the image measure $f(\nu)$ is the normal measure W^1_1 on ω_1. Otherwise, there exist disjoint sets $C, A' \subseteq \omega_1$ where C is c.u.b. and $f(\nu)(A') = 1$. If we define $f'\colon \omega_1 \to \omega_1$ by $f'(\alpha) =$ the largest element of $C \leq \alpha$, then $f' \circ f$ violates the minimality of f.

Fix $A_1 \subseteq A$ such that $\nu(A_1) = 1$ and $f \upharpoonright A_1$ is monotonically increasing and nonstrictly pressing down. For each $\gamma < \omega_1$ define

$$g(\gamma) = \sup\{\alpha \in A_1 : f(\alpha) \leq \gamma\}.$$

If $\alpha \in A_1$, then $g(f(\alpha)) \geq \alpha$. Also, note that $g(\gamma) < \omega_1$ since f is nonconstant and increasing. Let $x \in \omega^\omega$ and $C_0 \subseteq \omega_1$ be c.u.b. such that T^K_x is wellfounded, and for all $\gamma \in C_0$ we have $|T^K_x \upharpoonright \gamma| > g(\gamma)$. For each $\gamma \in C_0$ define B_γ by

$$\delta \in B_\gamma \iff \delta < \gamma \wedge \exists\alpha \in A_1\ (\gamma \leq \alpha \wedge \alpha = |T^K_x \upharpoonright \gamma(\delta)|). \tag{1}$$

Let $C_1 \subseteq C_0$ be the c.u.b. set from lemma 2.4 such that the sequence $\langle B_\gamma \subseteq \gamma : \gamma \in C_1\rangle$ coheres. Let $B = \bigcup_{\gamma \in C_1} B_\gamma$ and by lemma 2.1 there exists a good invariant strategy τ which codes B. By lemma 2.2, T^I_τ is wellfounded. Let c.u.b. $C_2 \subseteq C_1$ be closed under the function $\gamma \mapsto |T^I_\tau \upharpoonright \gamma|$. By lemma 2.3 there is $\sigma \in \omega^\omega$ such that T^K_σ is wellfounded and $C_\sigma \subseteq C_2$. Consider the simple set $S_{\tau,\sigma,x}$.

First, we claim that $S_{\tau,\sigma,x} \subseteq A$. Let $\alpha \in S_{\tau,\sigma,x}$. By definition of $S_{\tau,\sigma,x}$ there is some $\gamma \in C_\sigma$ and $\delta \in B$ such that $\delta < \gamma \leq \alpha$ and $\alpha = |T^K_x \restriction \gamma(\delta)|$. Since B is a union, then $\delta \in B_{\gamma'}$ for some $\gamma' \in C_1$. By the definition of C_1 we have $\delta \in B_\gamma$, and by equation 1 it follows that $\alpha \in A_1 \subseteq A$.

Second, we claim that $A_1 \cap f^{-1}C_\sigma \subseteq S_{\tau,\sigma,x}$. If $\alpha \in A_1 \cap f^{-1}C_\sigma$, then $\gamma \doteq f(\alpha) \in C_\sigma$. Since $\alpha \in A_1$ and $\gamma \in C_0$, then $\alpha \leq g(\gamma) < |T^K_x \restriction \gamma|$. So, for some $\delta \in B_\gamma$, $\alpha = |T^K_x \restriction \gamma(\delta)|$. Since $\gamma \in C_1$, then $\delta \in B$. Finally, since $\gamma \in C_\sigma$ and $C_\sigma \subseteq C_2$, we have satisfied the definition for $\alpha \in S_{\tau,\sigma,x}$.

The last two claims show that $S_{\tau,\sigma,x}$ is a simple set of ν measure one contained in A, a contradiction to the definition of ν. □

We can code subsets of ω_1 now by viewing each real $z \in \omega^\omega$ as coding a sequence of triples τ_n, σ_n, x_n and saying z codes the set $\phi(z) = \bigcup_n S_{\tau_n,\sigma_n,x_n}$. This is a $\mathbf{\Delta}^1_1$ coding of ω_1, namely:

Lemma 2.6 *For every $A \subseteq \omega_1$, there is a z with $\phi(z) = A$. Also, for every $\alpha < \omega_1$, $\{z \colon \alpha \in \phi(z)\} \in \mathbf{\Delta}^1_1$.*

Since we will carry out the analogous computation for functions in the next section (see lemma 3.5), we omit this proof.

3 Coding Functions

In the previous sections we showed how to code subsets of ω_1 by reals. In particular we showed that every subset of ω_1 is a countable union of simple sets. To show the strong partition relation on ω_1 we must code functions from ω_1 to ω_1. Of course, we could embed $\omega_1 \times \omega_1$ into ω_1 and view functions as subsets of ω_1 accordingly. The problem is that this coding does not seem to satisfy the conditions of Martin's theorem 1.2; a similar problem was encountered in the proof of [1]. Therefore, we will show in this section how the previous coding of subsets can be modified to a coding of functions which is good enough to get the strong partition relation on ω_1.

Let $\mathrm{WO}^{**} = \{z \in \omega^\omega : z_0 \in \mathrm{WO} \wedge z_1 \in 2^{\omega\times\omega}\}$. We think of $z \in \mathrm{WO}^{**}$ as coding a bounded subset A_z of $\omega_1 \times \omega_1$ by

$$(\alpha,\beta) \in A_z \iff \exists m,n \in \omega\ (|m|_{z_0} = \alpha \wedge |n|_{z_0} = \beta \wedge z_1(m,n) = 1).$$

We say $z, z' \in \omega^\omega$ are *compatible codes* if $z, z' \in \mathrm{WO}^{**}$ and

$$\forall m,m',n,n' < \omega\ (|m|_{z_0} = |m'|_{z'_0} \wedge |n|_{z_0} = |n'|_{z'_0}) \to z_1(m,n) = z'_1(m',n').$$

So, the predicate S representing incompatibility of codes is

$$\begin{aligned} S(z,z') \iff [&z \notin \mathrm{WO}^{**} \vee z' \notin \mathrm{WO}^{**} \vee \exists m,n,m',n' < \omega \\ &|m|_{z_0} = |m'|_{z'_0} \wedge |n|_{z_0} = |n'|_{z'_0} \wedge z_1(m,n) \neq z'_1(m',n')]. \end{aligned}$$

Clearly, $S \in \mathbf{\Sigma}^1_1$. Let T^S be a tree on ω^3 which projects to S. We again view every real τ as coding a strategy for II in an integer game and define invariance and good invariance for τ exactly as before. Again, we define a "cheap" coding of the subsets of $\omega_1 \times \omega_1$. Say τ *codes* $B \subseteq \omega_1 \times \omega_1$ if

$$\begin{aligned} &\forall x \in \mathrm{WO}\ [\tau(x) \in \mathrm{WO}^{**} \wedge |\tau(x)_0| \geq |x| \wedge \\ &\quad \forall \delta,\epsilon < |\tau(x)_0|\ \forall m,n < \omega\ (\delta = |m|_{\tau(x)_0} \wedge \epsilon = |n|_{\tau(x)_0}) \to \\ &\qquad ((\delta,\epsilon) \in B \leftrightarrow \tau(x)_1(m,n) = 1)]. \end{aligned}$$

The tree T^I attempting to witness a failure of good invariance is defined as before. A Solovay game again shows the cheap coding codes all subsets of $\omega_1 \times \omega_1$:

Lemma 3.1 *For all $B \subseteq \omega_1 \times \omega_1$ there is a good invariant strategy τ which codes B.*

We have the following analog to lemma 2.2.

Lemma 3.2 *For all $\tau \in \omega^\omega$ and for all $\alpha < \omega_1$, $T^I_\tau \restriction \alpha$ is wellfounded iff τ is a good $\leq \alpha$ invariant strategy.*

We now define a notion of *simple set* $S_{\tau,\sigma,x} \subseteq \omega_1 \times \omega_1$ coded by three reals $\tau, \sigma, x \in \omega^\omega$ by:

$$\begin{aligned}(\alpha,\beta) \in S_{\tau,\sigma,x} \iff & \exists \text{ limit } \gamma \leq \min(\alpha,\beta)\\ & \forall \delta < \gamma (|T^K_\sigma \restriction \delta| < \gamma \wedge |T^I_\tau \restriction \delta| < \gamma) \wedge\\ & \exists \delta, \epsilon < \gamma\, (\alpha = |T^K_x \restriction \gamma(\delta)| \wedge \beta = |T^K_x \restriction \gamma(\epsilon)| \wedge\\ & \quad \exists y \in \mathrm{WO}_{<\gamma} \exists m, n < \omega\, |m|_{\tau(y)_0} = \delta \wedge |n|_{\tau(y)_0} = \epsilon \wedge \tau(y)_1(m,n) = 1).\end{aligned}$$

This definition is the expected two dimensional analog of the definition of simple subset of ω_1, the exception being in the first line where γ is required to be at most $\min(\alpha,\beta)$, a more stringent requirement than $\gamma \leq \max(\alpha,\beta)$.

We also have the analog of lemma 2.4:

Lemma 3.3 *For each sequence $\langle B_\gamma \subseteq \gamma \times \gamma : \gamma \in C\rangle$ where $C \subseteq \omega_1$ is c.u.b., there is c.u.b. $D \subseteq C$ such that*

$$\forall \gamma_1, \gamma_2 \in D\ (\gamma_1 < \gamma_2 \to B_{\gamma_1} = B_{\gamma_2} \cap (\gamma_1 \times \gamma_1)).$$

We now prove the analog of lemma 2.5. The only real change is in the pressing down argument for ν.

Lemma 3.4 *Every function $h\colon \omega_1 \to \omega_1$ is a countable union of simple partial functions from ω_1 to ω_1.*

Proof. If not, let $h\colon \omega_1 \to \omega_1$ be a counterexample. Then the collection of subsets of h which are contained in countable unions of simple partial functions contained in h form a proper σ ideal on h. Let ν be a (nonprincipal) measure on h respecting this ideal, and let $f\colon \omega_1 \times \omega_1 \to \omega_1$ satisfy:

1. There is a ν measure one set $A_1 \subseteq \omega_1 \times \omega_1$ such that if $(\alpha, h(\alpha)), (\beta, h(\beta)) \in A_1$ with $\alpha \leq \beta$, then $f(\alpha, h(\alpha)) \leq f(\beta, h(\beta))$.
2. There does not exist a ν measure one set on which f is constant.
3. If $[f']_\nu < [f]_\nu$, then f' does not satisfy (1) and (2).

Since the map $(\alpha, h(\alpha)) \mapsto \alpha$ satisfies (1) and (2), f exists and ν almost everywhere satisfies $f(\alpha, h(\alpha)) \leq \alpha$. Note that the requirements on f are the same as in lemma 2.5 if we identify ν with a measure on ω_1 by identifying $(\alpha, h(\alpha))$ with α.

As before the image measure $f(\nu)$ is the normal measure on ω_1. Fix $A_1 \subseteq h$ such that $\nu(A_1) = 1$ and $f \restriction A_1$ is monotonically increasing on the first coordinate and nonstrictly pressing down. For each $\gamma < \omega_1$ define

$$g(\gamma) = \sup\{\max(\alpha, h(\alpha)) : (\alpha, h(\alpha)) \in A_1 \wedge f(\alpha, h(\alpha)) \leq \gamma\}.$$

If $(\alpha, h(\alpha)) \in A_1$, then $g(f(\alpha, h(\alpha))) \geq \max(\alpha, h(\alpha))$. Also, note that $g(\gamma) < \omega_1$ since f is nonconstant and increasing on the first coordinate. Let $x \in \omega^\omega$ and $C_0 \subseteq \omega_1$ be c.u.b. such that T^K_x is wellfounded, and for all $\gamma \in C_0$ we have $g(\gamma) < |T^K_x \restriction \gamma|$. For each $\gamma \in C_0$ define $B_\gamma \subseteq \gamma \times \gamma$ by

$$\begin{aligned}(\delta,\epsilon) \in B_\gamma \iff & \delta, \epsilon < \gamma \wedge \exists (\alpha, h(\alpha)) \in A_1\ (\gamma \leq \alpha\\ & \wedge \alpha = |T^K_x \restriction \gamma(\delta)| \wedge h(\alpha) = |T^K_x \restriction \gamma(\epsilon)|).\end{aligned} \tag{2}$$

Let $C_1 \subseteq C_0$ be the c.u.b. set from lemma 3.3 such that the sequence $\langle B_\gamma \subseteq \gamma \times \gamma : \gamma \in C_1\rangle$ coheres. Let $B = \bigcup_{\gamma \in C_1} B_\gamma$ and by lemma 3.1 there exists a good invariant strategy τ which codes B. By lemma 3.2 T^I_τ is wellfounded. Let c.u.b. $C_2 \subseteq C_1$ be closed under the function $\gamma \mapsto |T^I_\tau \restriction \gamma|$. By lemma 2.3 there is $\sigma \in \omega^\omega$ such that T^K_σ is wellfounded and $C_\sigma \subseteq C_2$. Consider the simple set $S_{\tau,\sigma,x}$.

First, we claim that $S_{\tau,\sigma,x} \subseteq h$. Let $(\alpha,\beta) \in S_{\tau,\sigma,x}$. By definition of $S_{\tau,\sigma,x}$ there is some $\gamma \in C_\sigma$ and $\delta, \epsilon \in B$ such that $\delta, \epsilon < \gamma \leq \alpha$ and $\alpha = |T^K_x \restriction \gamma(\delta)|$ and $\beta = |T^K_x \restriction \gamma(\epsilon)|$. Since B is an union, then

$(\alpha,\beta)\in B_{\gamma'}$ for some $\gamma'\in C_1$. By the definition of C_1 we have $(\delta,\epsilon)\in B_\gamma$, and by equation 2 it follows that $(\alpha,\beta)\in A_1\subseteq h$.

Second, we claim that $A_1\cap f^{-1}C_\sigma\subseteq S_{\tau,\sigma,x}$. If $(\alpha,h(\alpha))\in A_1\cap f^{-1}C_\sigma$, then $\gamma\doteq f(\alpha,h(\alpha))\in C_\sigma$. Note that $\gamma\leq\alpha$. Since $(\alpha,h(\alpha))\in A_1$ and $\gamma\in C_0$, then $\max(\alpha,h(\alpha))\leq g(\gamma)<|T^K_x\restriction\gamma|$. So, for some $(\delta,\epsilon)\in B_\gamma$, $\alpha=|T^K_x\restriction\gamma(\delta)|$ and $h(\alpha)=|T^K_x\restriction\gamma(\epsilon)|$. Since $\gamma\in C_1$, then $(\delta,\epsilon)\in B$. Finally, since $\gamma\in C_\sigma$ and $C_\sigma\subseteq C_2$, we have satisfied the definition for $(\alpha,h(\alpha))\in S_{\tau,\sigma,x}$.

We have shown that $S_{\tau,\sigma,x}$ is a simple subset of h of ν measure one, a contradiction to the definition of ν. □

Note that the proof of the previous lemma actually shows that every function $f\colon\omega_1\to\omega_1$ is of the form $f=\bigcup_p S_{\tau_p,\sigma_p,x_p}$ where all the trees $T^K_{\sigma_p}$, $T^K_{x_p}$, $T^I_{\tau_p}$ are wellfounded.

We are now able to give our coding of functions from ω_1 to ω_1 by reals. View each $z\in\omega^\omega$ as coding a countable sequence of triples τ_p,σ_p,x_p. Define $(\alpha,\beta)\in\phi(z)$ iff for some p, $(\alpha,\beta)\in S_{\tau_p,\sigma_p,x_p}$ and for all other p' and δ', $\epsilon'<\gamma'\in C_{\sigma'_p}$, if δ' lifts to α via $T_{x'_p}$ then ϵ' must lift to β. More precisely, we define

$$\begin{aligned}
&(\alpha,\beta)\in\phi(z)\iff\exists p<\omega\ \exists\text{ limit }\gamma\leq\min(\alpha,\beta)\\
&\quad\forall\delta<\gamma\ (|T^K_{\sigma_p}\restriction\delta|<\gamma\wedge|T^I_{\tau_p}\restriction\delta|<\gamma)\wedge\\
&\quad\exists\delta,\epsilon<\gamma\ |T^K_{x_p}\restriction\gamma(\delta)|=\alpha\wedge|T^K_{x_p}\restriction\gamma(\epsilon)|=\beta\wedge\\
&\qquad\exists y\in\mathrm{WO}_{<\gamma}\ \exists m,n<\omega\ |m|_{\tau_p(y)_0}=\delta\wedge|n|_{\tau_p(y)_0}=\epsilon\wedge\tau_p(y)_1(m,n)=1\wedge\\
&\quad\forall p'<\omega\ \forall\text{ limit }\gamma'\leq\min(\alpha,\beta)\ \forall y'\in\mathrm{WO}_{<\gamma'}\ \forall m',n'<\omega\\
&\quad[\forall\delta'<\gamma'\ (|T^K_{\sigma_{p'}}\restriction\delta'|<\gamma'\wedge|T^I_{\tau_{p'}}\restriction\delta'|<\gamma')\wedge\\
&\quad\exists\delta'<\gamma'\ |T^K_{x_{p'}}\restriction\gamma'(\delta')|=\alpha\wedge|m'|_{\tau_{p'}(y')_0}=\delta'\wedge\tau_{p'}(y')_1(m',n')=1]\to\\
&\qquad\quad\exists\epsilon'<\gamma'\ \ (\epsilon'=|n'|_{\tau_{p'}(y')_0}\wedge|T^K_{x_{p'}}\restriction\gamma'(\epsilon')|=\beta).
\end{aligned}$$

Lemma 3.4 and the following remark show that ϕ is a coding map for the functions from ω_1 to ω_1. Let R_α and $R_{\alpha,\beta}$ be the sets corresponding to the coding ϕ. To show this coding satisfies the hypotheses of theorem 1.2, first note that for any $\alpha,\beta<\omega_1$ that $z\in R_{\alpha,\beta}$ iff $\phi(z)(\alpha,\beta)$, since if $\phi(z)(\alpha,\beta)$ then necessarily β is uniquely determined.

The following lemma shows our coding satisfies condition 1 in definition 1.1.

Lemma 3.5 *For all $\alpha,\beta<\omega_1$, $R_{\alpha,\beta}\in\boldsymbol{\Delta}^1_1$.*

P r o o f. Fix $\alpha,\beta<\omega_1$. It is straightforward to compute that $R_{\alpha,\beta}\in\boldsymbol{\Delta}^1_1$ using the following observations. First, by a standard computation, for any tree T on $\omega\times\alpha$ with $\alpha<\omega_1$ and any $\beta,\gamma<\omega_1$, the set $\{x\colon|T(\beta)|=\gamma\}$ is $\boldsymbol{\Delta}^1_1$. Second, the quantifiers over $\mathrm{WO}_{<\gamma}$ in the above expression can be replaced by integer quantifiers. This is because these computations are relative to the set of x such that $\forall\delta<\gamma\ |T^I_x\restriction\delta|<\gamma$. Such x however are $<\gamma$ good and invariant. So, for any fixed γ we may fix a code w for γ whence the quantifications over $\mathrm{WO}_{<\gamma}$ reduce to integer quantification over the field of w. □

Lastly, the following lemma verifies condition 2 in definition 1.1.

Lemma 3.6 *If $\alpha<\omega_1$ and $A\subseteq R_\alpha$ is $\boldsymbol{\Sigma}^1_1$, then there exists $\beta'<\omega_1$ such that $\forall z\in A\ \exists\beta<\beta'$ $(z\in R_{\alpha,\beta})$.*

Proof. Fix $\alpha < \omega_1$. We will define a wellfounded relation $z \prec z'$ which is equivalent to $z, z' \in A \wedge \phi(z)(\alpha) < \phi(z')(\alpha)$. It is sufficient to show this relation is $\mathbf{\Sigma}^1_1$. We define

$$\begin{aligned}
z \prec z' \iff\ & z, z' \in A \wedge \exists p < \omega\ \exists \text{ limit } \gamma < \alpha \\
& \forall \delta < \gamma\ |T^K_{\sigma_p} \restriction \delta| < \gamma \wedge |T^I_{\tau_p} \restriction \delta| < \gamma) \wedge \\
& \exists \delta, \epsilon < \gamma\ |T^K_{x_p} \restriction \gamma(\delta)| = \alpha \wedge \\
& \quad \exists y \in \mathrm{WO}_{<\gamma} \exists m, n < \omega\ |m|_{\tau_p(y)_0} = \delta \wedge |n|_{\tau_p(y)_0} = \epsilon \wedge \tau_p(y)_1(m, n) = 1 \wedge \\
& \exists p' < \omega\ \exists \text{ limit } \gamma' < \alpha \\
& \forall \delta' < \gamma'\ (|T^K_{\sigma_{p'}} \restriction \delta'| < \gamma' \wedge |T^I_{\tau_{p'}} \restriction \delta'| < \gamma') \wedge \\
& \exists \delta', \epsilon' < \gamma'\ |T^K_{x_{p'}} \restriction \gamma'(\delta')| = \alpha \wedge \exists y' \in \mathrm{WO}_{<\gamma'}\ \exists m', n' < \omega \\
& |m'|_{\tau_{p'}(y')_0} = \delta' \wedge |n'|_{\tau_{p'}(y')_0} = \epsilon' \wedge \tau_{p'}(y')_1(m', n') = 1 \wedge \\
& \quad \text{“}|T^K_{x_p} \restriction \gamma(\epsilon)| < |T^K_{x_{p'}} \restriction \gamma'(\epsilon')|\text{”}.
\end{aligned}$$

Note that since $z, z' \in A \subseteq R_\alpha$, lines 1-4 say that $\phi(z)(\alpha) = |T^K_{x_p} \restriction \gamma(\epsilon)|$, and likewise, lines 5-8 say $\phi(z')(\alpha) = |T^K_{x_{p'}} \restriction \gamma'(\epsilon')|$. The last line in quotation marks represents a $\mathbf{\Sigma}^1_1$ predicate which checks $|T^K_{x_p} \restriction \gamma(\epsilon)| < |T^K_{x_{p'}} \restriction \gamma'(\epsilon')|$, given that $\epsilon \in \mathrm{wf}(T^K_{x_p} \restriction \gamma)$ and $\epsilon' \in \mathrm{wf}(T^K_{x_{p'}} \restriction \gamma')$. Thus, using our previous observations, this relation is $\mathbf{\Sigma}^1_1$, and hence has some countable length β'. Clearly, if $z \in A$, then $\phi(z)(\alpha) < \beta'$. □

All the conditions of theorem 1.2 have been satisfied, so we may conclude that

Theorem 3.7 *The strong partition relation on ω_1 holds.*

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