CHAPTER 7. CONFORMAL EXPANDING REPELLERS

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Conformal expanding repellers (abbreviation: CER's) were defined already in Chapter 5 and some basic properties of expanding sets and repellers in dimension 1 were discussed in Section 5.2. A more advanced geometric theory in the real 1-dimensional case was done in Section 5.6 ???. But now we have a new tool: Frostman Lemma and related facts from Chapter 6. Equipped with the theory of Gibbs measures and with the pressure function we are able to develop a geometric theory of CER's with Hausdorff measures and dimension playing the crucial role. We shall present this theory for $C^{1+\varepsilon}$ conformal expanding repellers in $I\!R^d$. Remind (Ch.5.2) that the assumed conformality forces for $d = 2$ that f is holomorphic or antiholomorphic and for $d \geq 3$ that f is locally a Möbius map. Conformality for $d = 1$ is meaningless, so we assume $C^{1+\varepsilon}$ in order to be able to rely on the Bounded Distortion for Iteration lemma.

We shall outline a theory of Gibbs measures from the point of view of multifractal spectra of dimensions (Sec.2) and pointwise fluctuations due to the Law of Iterated Logarithm (Sec.3) .

For $d = 2$ we shall apply this theory to study the boundary of the immediate basin of attraction to a sink for a rational mapping of the Riemann sphere in the case the basin is simply-connected and the mapping on the boundary is expanding, for example for the mapping $z \mapsto z^2 + c$ for $|c|$ small, for a quasicircle invariant under the action of a quasi-Fuchsian group ?????????????????????? and for the boundary of the "snowflake". In particular we study harmonic measure. We shall derive from this an information about the radial growth of the derivative of the Riemann mapping from the unit disc to the simply-connected domain under consideration.

Section 7.1.1. Pressure function and dimension.

Let $f : X \to X$ be a topologically mixing conformal expanding repeller in $I\!R^d$. As before we abbreviate notation of the pressure $\mathrm{P}(f, \phi)$, to $\mathrm{P}(\phi)$. We start with the following technical lemma.

Lemma 7.1.1. Let m be a Gibbs state (not necesserily invariant) on X and let $\phi : X \to I\!R$ be a Hölder continuous function. Assume $\mathrm{P}(\phi) = 0$. Then there is a constant $E \geq 1$ such that for all r small enough and all $x \in X$ there exists $n = n(x, r)$ such that

$$\frac{\log E + S_n\phi(x)}{-\log E - \log|(f^n)'(x)|} \leq \frac{\log m(B(x,r))}{\log r} \leq \frac{-\log E + S_n\phi(x)}{\log E - \log|(f^n)'(x)|}. \tag{7.1.0}$$

Proof. Take an arbitrary $x \in X$. Fix $r \in (0, C^{-1}\xi)$ and let $n = n(x, r) \geq 0$ be the largest integer so that

$$|(f^n)'(x)|rC \leq \xi, \tag{7.1.1}$$

where $C = C_{\mathrm{MD}}$ is the multiplicative distortion constant (corresponding to the Hölder continuous function $\log|f'|$), as in the Distortion Lemma for Iteration (Theorem 4.2.1), see Notation 5.2.2.??? Then

$$(7.1.2) \qquad f_x^{-n}(B(f^n(x),\xi)) \supset B(x,\xi|(f^n)'(x)|^{-1}C^{-1}) \supset B(x,r).$$

Now take n_0 such that $\lambda^{n_0-1} \ge C^2$. We then obtain

$$(7.1.3) \qquad |(f^{n+n_0})'|rC^{-1} \ge \xi.$$

Hence, again by the Distortion Lemma for Iteration

$$(7.1.4) \qquad f_x^{-n-n_0}(B(f^{n+n_0}(x),\xi)) \subset B\big(x,\xi|(f^{n+n_0})'(x)|^{-1}C\big) \subset B(x,r).$$

By the Gibbs property of the measure m, see (4.1.1), for a constant $E \ge 1$ (the constant C in (4.1.1)) we can write

$$E^{-1} \le \frac{\exp S_n\phi(x)}{m(f_x^{-n}(B(f^n(x),\xi)))} \quad \text{and} \quad \frac{\exp S_{n+n_0}\phi(x)}{m(f_x^{-(n+n_0)}(B(f^{n+n_0}(x),\xi)))} \le E.$$

Using this, (7.1.2), (7.1.4), the inequality $S_{n+n_0}\phi(x) \ge S_n\phi(x) + n_0 \inf\phi$, and finally increasing E so that the new $\log E$ is larger than the old $\log E - n_0\inf\phi$, we obtain

$$(7.1.5) \qquad \log E + S_n\phi(x) \ge \log m(B(x,r)) \ge -\log E + S_n\phi(x).$$

Using now (7.1.1) and (7.1.3), denoting $L = \sup|f'|$, and applying logarithms, we obtain

$$\frac{\log E + S_n\phi(x)}{\log|(f^n)'(x)|^{-1} - n_0\log L + \log\xi} \le \frac{\log m(B(x,r))}{\log r} \le \frac{-\log E + S_n\phi(x)}{\log|(f^n)'(x)|^{-1}\xi}.$$

Increasing further E so that $\log E \ge n_0\log L - \log\xi$, we can rewrite it in the "symmetric" form of (7.1.0). ♣

When we studied the pressure function $\phi \mapsto \mathrm{P}(\phi)$ in Chapters 2 and 4 the linear functional $\psi \mapsto \int \psi d\mu_\phi$ appeared. This was the Gateaux differential of P at ϕ (Theorem 2.5.5, Proposition 2.5.6 and (4.6.5')). Here the presence of an ambient smooth structure (1-dimensional or conformal) distingushes ψ's of the form $-t\log|f'|$. We obtain a link between the ergodic theory and the geometry of the embedding of X into $\mathbb{R}^d$.

Definition 7.1.2. Let μ be an ergodic f-invariant probability measure on X. Then by Birkhoff's Ergodic Theorem, for μ-almost every $x \in X$, the limit $\lim_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|$ exists and is equal to $\int\log|f'|d\mu$. We call this number the *Lyapunov characteristic exponent* of the map f with respect to the measure μ and we denote it by $\chi_\mu(f)$. In our case of expanding maps considered in this Chapter we obviously have $\chi_\mu(f) > 0$.

This definition does not demand the expanding property. It makes sense for an arbitrary invariant subset X of $\mathbb{R}^d$ or the Riemann sphere $\bar{\mathbb{C}}$, for f conformal (or differentiable in the real case) defined on a neighbourhood of X. There is no problem with the integrability because $\log|f'|$ is upper bounded on X. We do not exclude the possibility that $\chi_\mu = -\infty$. The notion of a Lyapunov characteristic exponent will play a crucial role also in subsequent chapters where non-expanding invariant sets will be studied.

Theorem 7.1.3. (Volume Lemma, expanding map, Gibbs measure case). Let m be a Gibbs state for a topologically mixing conformal expanding repeller $X \in I\!R^d$ and a Hölder continuous potential $\phi : X \to I\!R$. Then for m-almost every point $x \in X$ there exists the limit

$$\lim_{r\to 0} \frac{\log m(B(x,r))}{\log r}.$$

Moreover, this limit is almost everywhere constant and is equal to $\mathrm{h}_\mu(f)/\chi_\mu(f)$, where μ denotes the only f-invariant probability measure equivalent to m.

Proof. We can assume that $\mathrm{P}(\phi) = 0$. We can achieve it by subtracting $\mathrm{P}(\phi)$ from ϕ; the Gibbs measure class will stay the same (see Proposition 4.1.4). In view of the Birkhoff Ergodic Theorem, for μ-a.e $x \in X$ we have.

$$\lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \int \phi \, d\mu \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)| = \chi_\mu(f).$$

Combining these equalities with (7.8.0), along with the observation that $n = n(x,r) \to \infty$ as $r \to 0$, and using also the equality $\mathrm{h}_\mu(f) + \int \phi d\mu = \mathrm{P}(\phi) = 0$, we conclude that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)}.$$

The proof is finished. ♣

As an immediate consequence of this lemma and Corolaries 6.6.4 and 6.6.4a we get the following.

Theorem 7.1.4. If μ is a Gibbs state for a conformal expanding repeller $X \in I\!R^d$ and a Hölder continuous potential ϕ on X, then there exist Hausdorff and packing dimensions of μ and

$$\mathrm{HD}(\mu) = \mathrm{PD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f).$$

Using the above technique we can find a formula for the Hausdorff dimension and other dimensions of the whole set X. This is the solution of the non-linear problem, corresponding to the formula for Hausdorff dimension of the linear Cantor sets, discussed in the introduction. As f is Lipschitz continuous (or as f is forward expanding), the function

$$\mathrm{P}(t) := \mathrm{P}(-t \log|f'|)$$

is finite (see comments at the beginning of Section 2.5). As $|f'| \geq \lambda > 1$, it follows directly from the definition that $\mathrm{P}(t)$ is strictly decreasing from $+\infty$ to $-\infty$. In particular there exists exactly one parameter t_0 such that $\mathrm{P}(t_0) = 0$. We prove first the following.

Theorem 7.1.5. (Existence of geometric measures). Let t_0 be defined by $\mathrm{P}(t_0) = 0$. Write ϕ for $-t_0 \log |f'|$ restricted to X. Then each Gibbs state m corresponding to the function ϕ is a geometric measure with the exponent t_0. In particular $\lim_{r\to 0} \frac{\log m(B(x,r))}{\log r} = t_0$ for every $x \in X$.

Proof. We put in (7.1.0) $\phi = -t_0 \log |f'|$. Then using (7.1.1) (7.1.3) and $\sup |f'| \le L$ to replace $|(f^n)'(x)|^{-1}$ by r we obtain

$$\frac{\log E + t_0 \log r}{-\log E + \log r} \le \frac{\log m(B(x,r))}{\log r} \le \frac{-\log E + t_0 \log r}{\log E + \log r}$$

with a corrected constant E. Hence

$$\frac{\log E + t_0 \log r}{\log r} \le \frac{\log m(B(x,r))}{\log r} \le \frac{-\log E + t_0 \log r}{\log r} \tag{7.1.6}$$

for further corected E. In consequence

$$t_0 \le \frac{\log\big(m(B(x,r))/E\big)}{\log r} \quad \text{and} \quad \frac{\log\big(Em(B(x,r))\big)}{\log r} \le t_0,$$

hence

$$m(B(x,r))/E \le r^{t_0} \ \text{ and } \ Em(B(x,r)) \ge r^{t_0}.$$

(In the denominators we passed in Proof of Theorem 7.1.1 from r to $|(f^n)'(x)|^{-1}$ and here we passed back, so at this point the proof could be shortened. Namely we could deduce (7.1.6) directly from (7.1.5). However we needed to pass from $|(f^n)'(x)|^{-1}$ to r also in numerators and this point could not be simplified). ♣

As an immediate consequence of this theorem and Theorem 5.6.6 we get the following.

Corollary 7.1.6. The Hausdorff dimension of X is equal to t_0. Moreover it is equal to the packing and Minkowski dimensions. All Gibbs states corresponding to the potential $\phi = -t_0 \log |f'|$, as well as t_0-dimensional Hausdorff and packing measures are mutually equivalent with bounded Radon-Nikodym derivatives.

More on Volume Lemma. We end this section with a version of the Volume Lemma for a Borel probability invariant measure on the expanding repeller (X, f). In Chapter 9 we shall prove this without the expanding assumption assuming only positivity of Lyapunov exponent (though assuming also ergodicity) and the proof will be difficult. So we prove first a simpler version, which will be needed already in the next section. We start with a simple fact following from Lebesgue Theorem of differentiability a.e. ([Lojasiewicz, Th.7.1.4]) We provide a proof since it is very much in the spirit of Chapter 6.

Lemma 7.1.7. Every non-decreasing function $k : I \to I\!\!R$ defined on a bounded closed interval $I \subset I\!\!R$ is Lipschitz continuous at Lebesgue almost every point in I. In other words, for every $\varepsilon > 0$ there exist $L > 0$ and a set $A \subset I$ such that $|I \setminus A| < \varepsilon$, where $|\cdot|$ is the Lebesque measure in $I\!\!R$, and at each $r \in A$ the function k is Lipschitz continuous with the Lipschitz constant L.

Proof. Suppose on the contrary, that

$$B = \{x \in I : \sup\{y \in I : x \neq y,\ \frac{|k(x) - k(y)|}{|x - y|}\} = \infty\}$$

has positive Lebesgue measure. Write $I = [a, b]$. We can assume, by taking a subset, that B is compact and contains neither a nor b. For every $x \in B$ choose $x' \in I, x' \neq x$ such that

$$\frac{|k(x) - k(x')|}{|x - x'|} > 2\frac{k(b) - k(a)}{|B|}. \tag{7.1.6'}$$

Replace each pair x, x' by y, y' with $(y, y') \supset [x, x']$, and y, y' so close to x, x' that (7.1.6') still holds for y, y' instead of x, x'. In case when x or x' equals a or b we do not make the replacement.) We shall use for y, y' the old notation x, x' assuming $x < x'$. Now from the family of intervals (x, x') choose a finite family $\mathcal{I}$ covering our compact set B. From this family it is possible to choose a subfamily of intervals whose union still covers B and which consists of two subfamilies $\mathcal{I}^1$ and $\mathcal{I}^2$ of pairwise disjoint intervals.

Indeed. Start with $I_1 = (x_1, x_1') \in \mathcal{I}$ with minimal possible $x = x_1$ and maximal in $\mathcal{I}$ in the sense of inclusion. Having found $I_1 = (x_1, x_1'), ..., I_n = (x_n, x_n')$ we choose I_{n+1} as follows. Consider $\mathcal{I}_{n+1} := \{(x, x') \in \mathcal{I} : x \in \bigcup_{i=1,...,n} I_i, x' > \sup_{i=1,...,n} x_i'\}$. If $\mathcal{I}_{n+1}$ is non-empty, we set (x_{n+1}, x_{n+1}') so that $x_{n+1}' = \max\{x' : (x, x') \in \mathcal{I}_{n+1}\}$. If $\mathcal{I}_{n+1} = \emptyset$, we set (x_{n+1}, x_{n+1}') so that x_{n+1} is minimal possible to the right of $\max\{x_i' : i = 1, ..., n\}$ or equal to it, and maximal in $\mathcal{I}$. In this construction the intervals (x_n, x_n') with even n are pairwise disjoint, since each (x_{n+2}, x_{n+2}') has not been a member of $\mathcal{I}_{n+1}$. The same is true for odd n's. We define $\mathcal{I}^i$ for $i = 1, 2$ as the family of (x_n, x_n') for even, respectively odd, n. In view of the pairwise disjointness intervals of families $\mathcal{I}^1$ and $\mathcal{I}^2$, monotonicity of k and (7.1.6'), we get that

$$k(b) - k(a) \geq \sum_{n \in \mathcal{I}^1} k(x_n') - k(x_n) > 2\frac{k(b) - k(a)}{|B|} \sum_{n \in \mathcal{I}^1} (x_n' - x_n)$$

and the similar inequality for $n \in \mathcal{I}^2$. Hence, taking into account that $\mathcal{I}^1 \cup \mathcal{I}^2$ covers B, we get

$$2(k(b) - k(a)) > 2\frac{k(b) - k(a)}{|B|} \sum_{n \in \mathcal{I}^1 \cup \mathcal{I}^2} (x_n' - x_n) \geq 2\frac{k(b) - k(a)}{|B|}|B| = 2(k(b) - k(a)),$$

which is a contradiction finishing the proof. ♣

Corollary 7.1.8. For every Borel probability measure ν on a compact metric space (X, ρ) and for every $r > 0$ there exists a finite partition $\mathcal{P} = \{P_t, t = 1, ..., M\}$ of X into Borel sets of positive measure ν and with $\mathrm{diam}(\mathcal{P}) \leq r$ and there exists $C > 0$ such that for every $a > 0$

$$\nu(\partial_{\mathcal{P},a}) \leq Ca, \tag{7.1.7}$$

where $\partial_{\mathcal{P},a} := \bigcap_t \Big(\bigcup_{s \neq t} B(P_s, a)\Big)$.

Proof. Let $\{x_1, ..., x_N\}$ be a finite $r/4$-net in X. Fix $\varepsilon \in (0, r/4N)$. For each function $t \mapsto k_i(t) := \nu(B(x_i, t))$, $t \in I = [r/4, r/2]$, apply Lemma 7.1.7 and find appropriate L_i and A_i, for all $i = 1, ..., N$. Let $L = \max\{L_i, i = 1, ..., N\}$ and let $A = \bigcap_{i=1,...,N} A_i$. The set A has positive Lebesgue measure by the choice of ε. So, we can choose its point r_0 different from $r/4$ and $r/2$. Therefore, for all $a < a_0 := \min\{r_0 - r/4, r/2 - r_0\}$ and for all $i \in \{1, 2, \ldots, n\}$, we have $\nu(B(x_i, r_0 + a) \setminus B(x_i, r_0 - a)) \le 2La$. Hence, putting

$$\Delta(a) = \bigcup_i B(x_i, r_0 + a) \setminus B(x_i, r_0 - a)),$$

we get $\nu(\Delta(a)) \le 2LNa$. Define $\mathcal{P} = \{\bigcap_{i=1}^N B^{\kappa(i)}(x_i, r_0)\}$ as a family over all functions $\kappa : \{1, ..., N\} \to \{+, -\}$, where $B^+(x_i, r_0) := B(x_i, r_0)$ and $B^-(x_i, r_0) := X \setminus B(x_i, r_0)$, except κ yielding sets of measure 0, in particular except empty intersections. After removing from X of a set of measure 0, the partition $\mathcal{P}$ covers X. Since $r_0 \ge r/4$, the balls $B(x_i, r_0)$ cover X. Hence, for each non-empty $P_t \in \mathcal{P}$ at least one value of κ is equal to $+$. Hence $\operatorname{diam}(P_t) \le 2r_0 < r$. Note now that $\partial_{\mathcal{P},a} \subset \Delta(a)$. Indeed, let $x \in \partial_{\mathcal{P},a}$. Since $\mathcal{P}$ covers X there exists t_0 such that $x \in P_{t_0}$ so $x \notin P_t$ for all $t \ne t_0$. However, since $x \in \bigcup_{t \ne t_0} B(P_t, a)$, there exists $t_1 \ne t_0$ such that $\operatorname{dist}(x, P_{t_1}) < a$. Let $B = B(x_i, r_0)$ be such that $P_{t_0} \subset B^+$ and $P_{t_1} \subset B^-$, or *vice versa*. In the case when $x \in P_{t_0} \subset B^+$, by the triangle inequality $\rho(x, x_i) > r_0 - a$ and since $\rho(x, x_i) < r_0$, we get $x \in \Delta(a)$. In the case $x \in P_{t_0} \subset B^-$ we have $x \in B(x_i, r_0 + a) \setminus B(x_i, r_0) \subset \Delta(a)$. We conclude that $\nu(\partial_{\mathcal{P},a})) \le \nu(\Delta(a) \le 2LNa$ for all $a < a_0$. For $a \ge a_0$ it suffices to take $C \ge 1/a_0$. So the corollary is proved, with $C = \max\{2LN, 1/a_0\}$. ♣

Remark. If X is embedded for example in a compact manifold Y, then we can view ν as a measure on Y, we find a partition $\mathcal{P}$ of Y and then $\partial_{\mathcal{P},a} = B(\bigcup_{t=1,,,.M} \partial P_t, a)$, provided $M \ge 2$. This justifies the notation $\partial_{\mathcal{P},a}$.

Corollary 7.1.9. Let ν be a Borel probability measure on a compact metric space (X, ρ) andlet $f : X \to X$ be an endomorphism measurable with respect to the Borel σ-algebra on X and preserving measure ν. Le for every $r > 0$ let $\mathcal{P} = \{P_t, t = 1, ..., M\}$ be the partition of X constructed in Corollary 7.1.8. In particular $\operatorname{diam}(\mathcal{P} \le r$. Then for every $\delta > 0$ and ν-a.e. $x \in X$ there exists $n_0 = n_0(x)$ such that for every $n \ge n_0$

$$B(f^n(x), \exp(-n\delta)) \subset \mathcal{P}(f^n(x)) \tag{7.1.8}$$

Proof. Let $\mathcal{P}$ be the partition from Corollary 7.1.8. Fix an arbitrary $\delta > 0$. Then by Corollary 7.1.8

$$\sum_{n=0}^{\infty} \nu(\partial_{\mathcal{P},\exp(-n\delta)})) \le \sum_{n=0}^{\infty} C \exp(-n\delta) < \infty.$$

Hence by the f-invariance of ν, we obtain

$$\sum_{n=0}^{\infty} \nu(f^{-n}(\partial_{\mathcal{P},\exp(-n\delta)})) < \infty.$$

Applying now the Borel-Cantelli lemma for the family $\{f^{-n}(\partial_{\mathcal{P},\exp(-n\delta)})\}_{=1}^{\infty}$ we conclude that for ν-a.e $x \in X$ there exists $n_0 = n_0(x)$ such that for every $n \geq n_0$ we have $x \notin f^{-n}(\partial_{\mathcal{P},\exp(-n\delta)})$, so $f^n(x) \notin \partial_{\mathcal{P},\exp(-n\delta)}$. Hence, by the definition of $\partial_{\mathcal{P},\exp(-n\delta)}$, if $f^n(x) \in P$ for some $P \in \mathcal{P}$, then $f^n(x) \notin \bigcup_{s\neq t} B\big(P_s, \exp(-n\delta)\big)$. Thus

$$B(f^n(x), \exp -n\delta) \subset P.$$

We are done. ♣

Theorem 7.1.10. (Volume Lemma, expanding map, any measure case). Let ν be an f-invariant Borel probability measure on a topologically exact conformal expanding repeller (X, f), where $X \subset I\!R^d$.Then

$$\mathrm{HD}_*(\nu) \leq \frac{\mathrm{h}_\nu(f)}{\chi_\nu(f)} \leq \mathrm{HD}^*(\nu).$$

If in addition ν is ergodic, then

$$\mathrm{HD}(\nu) = \frac{\mathrm{h}_\nu(f)}{\chi_\nu(f)}.$$

Proof. Fix the partition $\mathcal{P}$ coming from Lemma 7.1.8 with $r = \min\{\xi, \eta\}$, where $\eta >$ was defined in (3.1.1). Then, as we saw in Chapter 4

$$\mathcal{P}^{n+1}(x) \subset f_x^{-n}(B(f^n(x), \xi)). \tag{7.1.10}$$

for every $x \in X$ and all $n \geq$. We shall work now to get a sort of opposite inclusion. Consider an arbitrary $\delta > 0$ and x so that (7.1.8) from Corollary 7.1.9 is satisfied for al $n \geq n_0(x)$. For every $0 \leq i \leq n$ define $k(i) = [i\frac{\delta}{\log\lambda} + \frac{\log\xi}{\log\lambda}] + 1$, $\lambda > 1$ being the expanding constant for $f : X \to X$ (see (3.1.1)). Hence $\exp(-i\delta) \geq \xi\lambda^{-k}$ and therefore $f^{-k}_{f^i(x)}(B(f^{i+k}(x), \xi)) \subset B(f^i(x), \exp -i\delta)$. So, using (7.1.8) for i in place of n, we get

$$f_x^{-(i+k)}(B(f^{i+k}(x), \xi)) \subset f_x^{-i}(\mathcal{P}(f^i(x))$$

for all $i \geq n_0(x)$. From this estimate for all $n_0 \leq i \leq n$, we conclude that

$$f_x^{-(n+k(n))}(B(f^{n+k(n)}(x), \xi) \subset \mathcal{P}_{n_0}^{n+1}(x).$$

Notice that for ν-a.e. x there is $a > 0$ such that $B(x, a) \subset \mathcal{P}^{n_0}(x)$, by the definition of $\partial_{\mathcal{P},\cdot}$. Therefore for all n large enough

$$f_x^{-(n+k(n))}(B(f^{n+k(n)}(x), \xi)) \subset \mathcal{P}^n(x). \tag{7.1.11}$$

It follows from (7.1.11) and (7.1.10) with $n + k(n)$ in place of n, that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} - \log\nu(\mathcal{P}^n(x)) &\leq \liminf_{n\to\infty} \frac{-\log\nu\big(f_x^{-(n+k(n))}(B(f^{n+k(n)}(x), \xi))\big)}{n} \\
&\leq \limsup_{n\to\infty} \frac{-\log\nu(f_x^{-(n+k(n))}(B(f^{n+k(n)}(x), \xi))}{n} \\
&\leq \lim_{n\to\infty} \frac{1}{n} - \log\nu(\mathcal{P}^n(x))(\mathcal{P}^{n+k(n)+1})(x).
\end{aligned}$$

The limits on the most left and most right-hand sides of these inequalities exist for ν-a.e. x by the Shennon-McMillan-Breiman Theorem (Theorem 1.5.4), see also (1.5.1), and their ratio is equal to 1. Letting $\delta \to 0$ we obtain the existence of the limit and the equality

$$\text{(7.1.12)}\quad \mathrm{h}_\nu(f,\mathcal{P},x) := \lim_{n\to\infty} \frac{1}{n} - \log \nu(\mathcal{P}^n(x))(\mathcal{P}^n)(x) = \lim_{n\to\infty} \frac{-\log \nu(f_x^{-n}(B(f^n(x),\xi))}{n}.$$

In view of Birkhoff's Ergodic Theorem, thelimit

$$\text{(7.1.13)}\qquad \chi_\nu(f,x) := \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|,$$

exists for ν-a.e. $x \in X$. Dividing side by side (7.1.12) by (7.1.13) and using (7.1.1)-(7.1.4), we get

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = \frac{\mathrm{h}_\nu(f,\mathcal{P},x)}{\chi_\nu(f,x)}.$$

Since by the Shennon-McMillan-Breiman Theorem, and Birkhoff's Ergodic Theorem,

$$\frac{\int \mathrm{h}_\nu(f,\mathcal{P},x)\,d\nu(x)}{\int \chi_\nu(f,x)\,d\nu} = \frac{\mathrm{h}_\nu(f,\mathcal{P})}{\chi_\nu(f)}. = \frac{\mathrm{h}_\nu(f)}{\chi_\nu(f)},$$

where the latter equality was written since f is expansive and $\mathrm{diam}(\mathcal{P})$ is less than the expansiveenss constant of $f : X \to X$ which at least excesds η, there thus exists a positive measure set where $\frac{\mathrm{h}_\nu(f,\mathcal{P},x)}{\chi_\nu(f,x)} \le \frac{\mathrm{h}_\nu(f)}{\chi_\nu(f)}$ and a positive measure set where the opposite inequality holds. Therefore

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \le \frac{\mathrm{h}_\nu(f)}{\chi_\nu(f)}$$

and the opposite inequality also holds on a positive measure set. In view of definitions of HD_* and HD^* and by Corollary 6.6.4, this finishes the proof of the first part of our Theorem. In the ergodic case $\mathrm{h}_\nu(f,\mathcal{P},x) = \mathrm{h}_\nu(f)$ and $\chi_\nu(f,x) = \chi_\nu(f)$ for ν-a.e. $x \in X$. So

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = \frac{\mathrm{h}_\nu(f)}{\chi_\nu(f)}$$

and we are done in this case as well. ♣

Section 7.8. Multifractal analysis of Gibbs state.

In the previous section we linked to a (Gibbs) measure only one dimension number, $\mathrm{HD}(m)$. Here one of our aims is to introduce 1-parameter families of dimensions, so-called spectra of dimensions. In these definitions we do not need the mapping f. Let ν be a Borel probability measure on a metric space X. Recall from Chapter 6.7 that given $x \in X$ we defined the lower and upper pointwise dimension of ν at x by putting respectively

$$\underline{d}_\nu(x) = \liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \text{ and } \overline{d}_\nu(x) = \limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}.$$

If $\underline{d}_\nu(x) = \overline{d}_\nu(x)$, we call the common value the pointwise dimension of ν at x and we denote it by $d_\nu(x)$. The function d_ν is called the *dimension spectrum* of the measure ν. For any $\alpha \leq 0 \leq \infty$ write

$$X_\nu(\alpha) = \{x \in X : d_\nu(x) = \alpha\}.$$

The domain of d_ν namely the set $\bigcup_\alpha X_\nu(\alpha)$ is called *a regular part* of X and its complement $\hat{X}$ a *singular part.* The decomposition of the set X as

$$X = \bigcup_{0\leq\alpha\leq\infty} X_\nu(\alpha) \cup \hat{X}.$$

is called the *multifractal decomposition with respect to the dimension spectrum.*

Define the $F_\nu(\alpha)$-*spectrum for dimensions* function related to Hausdorff dimension by

$$F_\nu(a) = \mathrm{HD}(X_\nu(\alpha)),$$

where we define the domain of F_ν as $\{\alpha : X_\nu(\alpha) \neq \emptyset\}$.

Note that by Theorem 7.1.5 if (X, f) is a topologically exact expanding conformal repeller and $\nu = \mu_{-\mathrm{HD}(X)\log|f'|}$ then all $X_\nu(\alpha)$ are empty except $X_\nu(\mathrm{HD}(X))$. In particular the domain of F_ν is in this case just one point $\mathrm{HD}(X)$.
Let for every real $q \neq 1$

$$R_q(\nu) := \frac{1}{q-1} \lim_{r\to 0} \frac{\log \sum_{i=1}^N \nu(B_i)^q}{\log r},$$

where $N = N(r)$ is the total number of boxes B_i of the form $B_i = \{(x_1, ..., x_d) \in I\!R^d : rk_j \leq x_j \leq r(k_j + 1),\ j = 1, ..., d\}$ for integers $k_j = k_j(i)$ such that $\nu(B_i) > 0$. This function is called *Rényi spectrum for dimensions*, provided the limit exists. It is easy to check (exercise 7.2.1) that it is equal to the *Hentschel-Procaccia spectrum*

$$HP_q(\nu) := \frac{1}{q-1} \lim_{r\to 0} \frac{\log \inf_{\mathcal{G}_r} \sum_{B(x_i,r)\in\mathcal{G}_r} \nu(B(x_i, r))^q}{\log r},$$

where infimum is taken over all $\mathcal{G}_r$ being finite or countable coverings of the (topological) support of ν by balls of radius r centered at $x_i \in X$, or

$$HP_q(\nu) := \frac{1}{q-1} \lim_{r\to 0} \frac{\log \int_X \nu(B(x, r))^{q-1} d\nu(x)}{\log r}$$

provided the limits exist. For $q = 1$ we define the *information dimension* $I(\nu)$ as follows. Set

$$H_\nu(r) = \inf_{\mathcal{F}_r}\Big(- \sum_{B\in\mathcal{F}_r} \nu(B) \log \nu(B)\Big),$$

where infimum is taken over all partitions $\mathcal{F}_r$ of a set of full measure ν into Borel sets B of diameter at most r. We define

$$I(\nu) = \lim_{r\to 0} \frac{H_\nu(r)}{-\log r}$$

provided the limit exists. A complement to Corollary 6.6.4 is that

$$\mathrm{HD}_*(\nu) \le I(\nu) \le \mathrm{PD}^*(\nu). \tag{7.8.0}$$

For the proof see Exercise 7.2.5.Note that for Rényi and HP dimensions it does not make any difference whether we consider coverings of the topological support (the smallest closed set of full measure) of a measure or any set of full measure, since all balls have the same radius r, so we can always choose locally finite (number independent of r) subcovering. These are "box type" dimension quantities.

A priori there is no reason for the function $F_\nu(\alpha)$ to behave nicely. If ν is an f-invariant ergodic measure for (X, f), a topologically exact conformal expanding repeller, then at least we know that for $\alpha_0 = \mathrm{HD}(\nu)$, we have $d_\nu(x) = \alpha_0$ for ν-a.e. x (by the Volume Lemma: 7.1.3 and Theorem 7.1.4 for a Gibbs measure ν of a Hölder continuous function and by Theorem 7.1.10 in the general case). So, in particular we know at least that the domain of $F_\alpha(\nu)$ is nonempty. However for $\alpha \ne \alpha_0$ we have then $\nu(X_\nu(\alpha)) = 0$ so $X_\nu(\alpha)$ are not visible for the measure ν. Whereas the function $HP_q(\nu)$ can be determined by statistical properties of ν-typical (a.e.) trajectory, the function $F_\nu(\alpha)$ seems intractable. However if $\nu = \mu_\phi$ is an invariant Gibbs measure for a Hölder continuous function (potential) ϕ, then miraculously the above spectra of dimensions happen to be real-analytic functions and $-F_{\mu_\phi}(-p)$ and $HP_q(\mu_\phi)$ are mutual Legendre transforms. Compare this with the pair of Legendre-Fenchel transforms: pressure and -entropy, Remark 2.5.3. Thus fix an invariant Gibbs measure μ_ϕ corresponding to a Hölder continuous potential ϕ. We can assume without loosing generality that $\mathrm{P}(\phi) = 0$. Indeed, starting from an arbitrary ϕ, we can achieve this without changing μ_ϕ by subtracting from ϕ its topological pressure (as at the beginning of the proof of Lemma 7.1.3). Having fixed ϕ, in order to simplify notation, we denote $X_{\mu_\phi}(\alpha)$ by X_α and F_{μ_ϕ} by F. We define a two-parameter family of auxiliary functions $\phi_{q,t} : X \to I\!R$ for $q, t \in I\!R$, by setting

$$\phi_{q,t} = -t \log |f'| + q\phi.$$

Lemma 7.8.1. For every $q \in I\!R$ there exists a unique $t = T(q)$ such that $\mathrm{P}(\phi_{q,t}) = 0$.

Proof. This lemma follows from the fact the function $t \mapsto \mathrm{P}(\phi_{q,t})$ is decreasing from ∞ to $-\infty$ for every q (see comments preceding Theorem 7.1.5 and at the beginning of Section 2.5) and the Darboux theorem. ♣

We deal with invariant Gibbs measures $\mu_{\phi_{q,T(q)}}$ which we denote for abbreviation by μ_q and with the measure μ_ϕ so we need to know a relation between them. This is explained in the following.

Lemma 7.8.2. For every $q \in I\!\!R$ there exists $C > 0$ such that for all $x \in X$ and $r > 0$

$$C^{-1} \leq \frac{\mu_q(B(x,r))}{r^{T(q)}\mu_\phi(B(x,r))^q} \leq C. \tag{7.8.1}$$

Proof. Let $n = n(x,r)$ be defined as in Lemma 7.1.1. Then, by (7.1.5), (7.1.1) and (7.1.3), the ratios

$$\frac{\mu_\phi(B(x,r))}{\exp S_n\phi(x)}, \quad \frac{\mu_q(B(x,r))}{|(f^n)'(x)|^{-T(q)}\exp qS_n\phi(x)}, \quad \frac{r}{|(f^n)'(x)|^{-1}}$$

are bounded from below and above by positive constants independent of x, r. This yields the estimates (7.8.1) ♣

Let us prove the following.

Lemma 7.8.3. For any f-invariant ergodic probability measure τ on X and for τ-a.e. $x \in X$ we have

$$d_{\mu_\phi}(x) = \frac{\int \phi d\tau}{-\int \log|f'|d\tau}.$$

Proof. Using formula (7.1.0) in Lemma 7.1.1 and Birkhoff 's Ergodic Theorem, we get

$$d_{\mu_\phi}(x) = \lim_{n\to\infty} \frac{S_n\phi(x)}{\log|(f^n)'(x)|^{-1}} = \frac{\lim_{n\to\infty}\frac{1}{n}S_n\phi(x)}{\lim_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|^{-1}} = \frac{\int\phi d\tau}{-\int\log|f'|d\tau}.$$

♣

One can conclude from this, that the singular part $\hat{X}$ of X has zero measure for every f-invariant τ. Yet the set $\hat{X}$ is usually big, see Exercise 7.2.4.

On the Legendre transform. Let $k = k(q) : I \to I\!\!R$ be a convex function on $I = [\alpha_1(k), \alpha_2(k)]$ where $-\infty \leq \alpha_1(k) \leq \alpha_2(k) \leq \infty$ (i.e. I is either a point or a closed interval or a semiline or $I\!\!R$). The Legendre transform of k is the function g of a new variable p defined by

$$g(p) = \sup_{q\in I}\{pq - k(q)\}$$

everywhere where a finite supremum exists. It can be easily proved (Exercise 7.2.2) that the domain of g is also either a point, or a closed interval or a semiline or $I\!\!R$. It is also easy to show that g is convex and that the Legendre transform is involutive. We then say that the functions k and g form a Legendre transform pair.

Proposition 7.8.4. If two convex functions k and g form a Legendre transform pair then $g(k'(q)) = qk'(q) - k(q)$, where $k'(q)$ is any number between the left and right hand side derivative of k at q, which are defined as $-\infty, \infty$ at $\alpha_1(k)$, $\alpha_2(k)$ respectively, if these end points are finite. We set $0 \cdot \pm\infty = 0$ in case $k' = \pm\infty$ at $q = \alpha_i(k) = 0$. If

$\alpha_2(k) = \infty$ (similarly if $\alpha_1(k) = -\infty$), then for $k'(\infty)$ defined as $\lim_{q\to\infty} k'(q)$, it holds $g(k'(\infty)) = \lim_{q\to\infty} g(k'(q))$.

Note that if k is C^2 with $k'' > 0$, therefore strictly convex, then also $g'' > 0$ at all points $k'(q)$ for $\alpha_1(k) < q < \alpha_2(k)$, therefore g is strictly convex on $[k'(\alpha_1(k)), k'(\alpha_2(k))]$. Outside this interval g is affine in its domain. If the domain of k is one point then g is affine on $I\!R$ and *vice versa.*

We are now in position to formulate our main theorem in this section gathering in particular some facts already proven.

Theorem 7.8.5.

(a) The pointwise dimension $d_{\mu_\phi}(x)$ exists for μ_ϕ-almost every $x \in X$ and

$$d_{\mu_\phi}(x) = \frac{\int \phi d\mu_\phi}{-\int \log|f'| d\mu_\phi} = \mathrm{HD}(\mu_\phi) = \mathrm{PD}(\mu_\phi).$$

(b) The function $q \mapsto T(q)$ for $q \in I\!R$, is real analytic, $T(0) = \mathrm{HD}(X)$, $T(1) = 0$, $T'(q) = \frac{\int \phi d\mu_q}{\int \log|f'| d\mu_q} < 0$ and $T''(q) \geq 0$.

(c) For all $q \in I\!R$ we have $\mu_q(X_{-T'(q)}) = 1$, where μ_q is the invariant Gibbs measure for the potential $\phi_{q,T(q)}$, and $\mathrm{HD}(\mu_q) = \mathrm{HD}(X_{-T'(q)})$.

(d) For every $q \in I\!R$, $F(-T'(q)) = T(q) - qT'(q)$, i.e. $p \mapsto -F(-p)$ is Legendre transform of $T(q)$.
If the measures μ_ϕ and $\mu_{-\mathrm{HD}(X)\log|f'|}$ (the latter discussed in Theorem 7.1.5 and Corollary 7.1.6) do not coincide, then $T'' > 0$ and $F'' < 0$, i.e. the functions T and F are respectively strictly convex on $I\!R$, and stricctly concave on $[-T'(\infty), -T'(-\infty)]$ which is a bounded interval in $I\!R^+ = \{\alpha \in I\!R : \alpha > 0\}$. If $\mu_\phi = \mu_{-\mathrm{HD}(X)\log|f'|}$ then T is affine and the domain of F is one point $-T'$.

(e) For every $q \neq 1$ the HP and Rényi spectra exist (i.e. limits in the definitions exist) and $\frac{T(q)}{1-q} = HP_q(\mu_\phi) = R_q(\mu_\phi)$. For $q = 1$ the information dimension $I(\mu_\phi)$ exists and

$$\lim_{q\to 1, q\neq 1} \frac{T(q)}{1-q} = -T'(1) = \mathrm{HD}(\mu_\phi) = \mathrm{PD}(\mu_\phi) = I(\mu_\phi).$$

Insert Figures: graph of T, graph of F [Pesin, p.219, 220]

Proof. **1.** Since $\mathrm{P}(\phi) = 0$, the part (a) is an immediate consequence of Lemma 7.1.3 and its second and third equalities follow from Theorem 7.1.4. The first equality is also a special case of Lemma 7.8.3 with $\tau = \mu_\phi$.

2. We shall prove some statements of the part (b). The function $\phi_{q,t} = -t\log|f'| + q\phi$, from $I\!R^2$ to $C^\theta(X)$, where θ is a Hölder exponent of the function ϕ, is affine. Since by a result of Ruelle (see [Ru])??????? the topological pressure function $\mathrm{P} : C^\theta \to I\!R$ is real analytic, then the composition which we denote $\mathrm{P}(q,t)$ is real analytic. Hence the real analyticity of $T(q)$ follows immediately from the Implicit Function Theorem once we verify the non-degeneracy assumption. In fact C^2-smoothness of $\mathrm{P}(q,t)$ is sufficient to proceed further (here only C^1), which has been proved in Theorem 4.7.4. Indeed, due to Theorem 4.6.5 for every $(q_0,t_0)\in I\!R^2$

$$\frac{\partial \mathrm{P}(q,t))}{\partial t}|_{(q_0,t_0)} = -\int_X \log|f'|\,d\mu_{q_0,t_0} < 0, \tag{7.8.2}$$

where μ_{q_0,t_0} is the invariant Gibbs state of the function ϕ_{q_0,t_0}. Differentiating with respect to q the equality $\mathrm{P}(q,t)=0$ we obtain

$$0 = \frac{\partial \mathrm{P}(q,t)}{\partial t}|_{(q,T(q))}\cdot T'(q) + \frac{\partial \mathrm{P}(q,t))}{\partial q}|_{(q,T(q))} \tag{7.8.3}$$

hence we obtain the standard formula

$$T'(q) = -\frac{\partial \mathrm{P}(q,t))}{\partial q}|_{(q,T(q))}\Big/\frac{\partial \mathrm{P}(q,t)}{\partial t}|_{(q,T(q))},$$

Again using (4.6.5') and $\mathrm{P}(\phi_{q,T(q)})=0$, we obtain

$$T'(q) = \frac{\int \phi\, d\mu_q}{\int \log|f'|\,d\mu_q} \le \frac{-\mathrm{h}_{\mu_q}(f)}{\int \log|f'|\,d\mu_q} < 0, \tag{7.8.4}$$

the latter true since the entropy of any invariant Gibbs measure for Hölder function is positive, see for example Theorem 4.2.7. The equality $T(0)=\mathrm{HD}(X)$ is just Corollary 7.1.2. $T(1)=0$ follows from the equality $\mathrm{P}(\phi)=0$.

3. The inequality $T''(q)\ge 0$ follows from the convexity of $\mathrm{P}(q,t)$, see Theorem 2.5.2. Indeed the assumption that the part of $I\!R^3$ above the graph of $\mathrm{P}(q,t)$ is convex implies that its intersection with the plane (q,t) is also convex. Since $\frac{\partial \mathrm{P}(q,t))}{\partial t}|_{(q_0,t_0)}<0$, this is the part of the plane above the graph of T. Hence T is a convex function. We avoided in the above consideration an explicit computation of T''. However to discuss strict convexity (a part of (d)) it is necessary to compute it. Differentiating (7.8.3) with respect to q we obtain the standard formula

$$T''(q) = \frac{T'(q)^2\frac{\partial^2 \mathrm{P}(q,t)}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}(q,t)}{\partial q\partial t} + \frac{\partial^2 \mathrm{P}(q,t)}{\partial q^2}}{-\frac{\partial \mathrm{P}(q,t)}{\partial t}} \tag{7.8.5}$$

with the derivatives of P taken at $(q,T(q))$. The numerator is equal to

$$\Big(T'(q)\frac{\partial}{\partial t}+\frac{\partial}{\partial q}\Big)^2\mathrm{P}(q,t) = \sigma^2_{\mu_q}(-T'(q)\log|f'|+\phi)$$

by Theorem 4.7.4, since this is the second order derivative of $P: C(X)\to I\!R$ in the direction of the function $-T'(q)\log|f'|+\phi$.

The inequality $\sigma^2 \geq 0$, true by definition, implies $T'' \geq 0$ since the denominator in (7.8.5) is positive by (7.8.2).

By Theorem 1.11.3 $\sigma^2_{\mu_q}(-T'(q)\log|f'| + \phi) = 0$ if and only if the function $-T'(q)\log|f'|+\phi$ is cohomologous to a constant, say to a. It follows then from the equality in (7.8.4) that $a = \int a\, d\mu_q = \int(-T'(q)\log|f'| + \phi)d\mu_q = 0$. Therefore $T'(q)\log|f'|$ is cohomologous to ϕ and, as $\mathrm{P}(\phi) = 0$, also $\mathrm{P}(T'(q)\log|f'|) = 0$. Thus, by Theorem 7.1.5 and Corollary 7.1.2, $T'(q) = -\mathrm{HD}(X)$ and consequently ϕ is cohomologous to the function $-\mathrm{HD}(X)\log|f'|$. This implies that $\mu_\phi = \mu_{-\mathrm{HD}(X)\log|f'|}$, the latter being the equilibrium (invariant Gibbs) state of the potential $-\mathrm{HD}(X)\log|f'|$. Therefore, in view of our formula for T'', if $\mu_\phi \neq \mu_{-\mathrm{HD}(X)\log|f'|}$, then $T''(q) > 0$ for all $q \in \mathbb{R}$.

4. We prove (c). By Lemma 7.8.3 applied to $\tau = \mu_q$, there exists a set $\tilde{X}_q \subset X$, of full measure μ_q, such that for every $x \in \tilde{X}_q$ there exists

$$d_{\mu_\phi}(x) = \lim_{r\to 0}\frac{\log\mu_\phi(B(x,r))}{\log r} = \frac{\int \phi d\mu_q}{-\int \log|f'|d\mu_q} = -T'(q).$$

the latter proved in (b). Hence $\tilde{X}_q \subset X_{-T'(q)}$. Therefore $\mu_q(X_{-T'(q)}) = 1$. By Lemma 7.8.2 for every $B = B(x,r)$

$$|\log\mu_q(B) - T(q)\log r - q\log\mu_\phi(B)| < C$$

for some constant $C \in \mathbb{R}$. Hence

$$\left|\frac{\log\mu_q(B)}{\log r} - T(q) - q\frac{\log\mu_\phi(B)}{\log r}\right| \to 0 \tag{7.8.6}$$

as $r \to 0$. Using (7.8.6), observe that for every $x \in X_{-T'(q)}$, in particular for every $x \in \tilde{X}_q$,

$$\lim_{r\to 0}\frac{\log\mu_q(B)}{\log r} = T(q) + q\lim_{r\to 0}\frac{\log\mu_\phi(B)}{\log r} = T(q) - qT'(q).$$

Although $\tilde{X}_q$ can be much smaller than $X_{-T'(q)}$, miraculously their Hausdorff dimensions coincide. Indeed the measure μ_q restricted to either $\tilde{X}_q$ or to $X_{-T'(q)}$ satisfies the assumptions of Theorem 6.6.3 with $\theta_1 = \theta_2 = T(q) - qT'(q)$. Therefore

$$\mathrm{HD}(\tilde{X}_q) = \mathrm{HD}(X_{-T'(q)}) = T(q) - qT'(q) \tag{7.8.7}$$

and consequently

$$F(-T'(q)) = T(q) - qT'(q).$$

Remarks. (a) If we take a set larger than $X_{-T'(q)}$, namely replacing in the definition of $X_\nu(\alpha)$ the dimension d_ν by the lower dimension $\underline{d}_\nu$ we still obtain the same Hausdorff dimension, again by Theorem 6.6.3.

(b) Some authors replace in the definition of $X_\nu(\alpha)$ the value $d_\nu(x)$ by $\underline{d}_\nu(x)$. Then there is no singular part. In view of a) the $F_\nu(\alpha)$ spectrum is the same for $\nu = \mu_\phi$.

(c) Notice that (7.8.7) means that $\mathrm{HD}(X_{-T'(q)})$ is the value where the straight line tangent to the graph of T at $(q, T(q))$ intersects the range axis.

In the next steps of the proof the following will be useful.

Claim. (Variational Principle fo T.) For any f-invariant ergodic probability measure τ on X, consider the following linear equation of variables q, t

$$\int \phi_{q,t} d\tau + \mathrm{h}_\tau(f) = 0$$

that is

(7.8.8) $$t = t_\tau(q) = \frac{\mathrm{h}_\tau(f)}{\int \log|f'| d\tau} + q \frac{\int \phi d\tau}{\int \log|f'| d\tau}.$$

Then for every $q \in I\!R$

$$T(q) = \sup_\tau \{t_\tau(q)\} = t_{\mu_q}(q),$$

Where the supremum is taken over all f-invariant ergodic probability measures τ on X.

Proof of the Claim. Since $\int \phi_{q,t} d\tau + \mathrm{h}_\tau(f) \le \mathrm{P}(\phi_{q,t})$ and since $\frac{\partial \mathrm{P}(q,t)}{\partial t} < 0$ (compare the proof of convexity of T), we obtain

$$t_\tau(q) \le T(q).$$

On the other hand by (7.8.8), and using $\mathrm{P}(\phi_{q,T(q)} = 0$, we obtain

$$t_{\mu_q}(q) = \frac{\mathrm{h}_{\mu_q}(f) + q\int \phi d\mu_q}{\int \log|f'| d\mu_q} = \frac{T(q)\int \log|f'| d\mu_q}{\int \log|f'| d\mu_q} = T(q).$$

The Claim is proved. ♣

5. We continue Proof of Theorem 7.8.5. We shall prove the missing parts of (d). We have already proved ithat

$$F(-T'(q)) = \mathrm{HD}(X_{-T'(q)}) = \mathrm{HD}(\mu_q) = T(q) - qT'(q).$$

Note that $[-T'(\infty), -T'(-\infty)] \subset I\!R^+ \cup \{0, \infty\}$ since $T'(q) < 0$ for all q. Note finally that

$$-T'(-\infty) = \lim_{q \to -\infty} \frac{-\int \phi d\mu_q}{\int \log|f'| d\mu_q} \le \frac{\sup(-\phi)}{\inf \log|f'|} < \infty$$

and

$$-T'(\infty) = \lim_{q \to \infty} \frac{-\int \phi d\mu_q}{\int \log|f'| d\mu_q}.$$

The expressions under lim are positive, (see (7.8.4)). It is enough now to prove that they are bounded away from 0 as $q \to \infty$. To this end choose q_0 such that $T(q_0) < 0$. By our Claim (Variational Principle for T) $t_{\mu_q}(q_0) \le T(q_0)$. Since $t_{\mu_q}(0) \ge 0$, we get

$$-q_0 \frac{\int \phi d\mu_q}{\int \log |f'| d\mu_q} = t_{\mu_q}(0) - t_{\mu_q}(q_0) \ge |T(q_0)|.$$

Hence $\frac{-\int \phi d\mu_q}{\int \log |f'| d\mu_q} \ge |T(q_0)|/q_0 > 0$ for all q.

6. To end the proof of (d) we need to prove the formula for F at $-T'(\pm\infty)$ (in case T is not affine) and to prove that for $\alpha \notin [-T'(\infty), -T'(-\infty)]$ the sets $X_{\mu_\phi}(\alpha)$ are empty. First note the following.

6a. For any f-invariant ergodic probability measure τ on X, there exists $q \in I\!R \cup \{\pm\infty\}$ such that

$$\frac{\int \phi d\tau}{\int \log |f'| d\tau} = \frac{\int \phi d\mu_q}{\int \log |f'| d\mu_q}. \tag{7.8.9}$$

($\lim_{q\to\pm\infty}$ in the $\pm\infty$ case).

Indeed, by the Claim the graphs of the functions $t_\tau(q)$ and $T(q)$ do not intersect transversally (they can be only tangent) and hence the first graph which is a straight line, is parallel to a tangent to the graph of T at a point $(q_0, T(q_0)$, or one of its asymptots, at $-\infty$ or $+\infty$. Now (7.8.9) follows from the same formula (7.8.8) for $\tau = \mu_{q_0}$, since the graph of $t_{\mu_{q_0}}$ is tangent to the graph of T just at $(q_0, T(q_0))$. (Note that the latter sentence proves the formula $T'(q) = \frac{\int \phi d\mu_q}{\int \log |f'| d\mu_q}$ in a different way than in 2, namely *via* Variational Principle for T.).

6b. Proof that $X_\alpha = \emptyset$ for $\alpha \notin [-T'(\infty), -T'(-\infty)]$. Suppose there exists $x \in X$ with $\alpha := d_{\mu_\phi}(x) \notin [-T'(\infty), -T'(-\infty)]$. Consider any sequence of integers $n_k \to \infty$ and real numbers b_1, b_2 such that

$$\lim_{k\to\infty} \frac{1}{n_k} S_n \phi(x) = b_1, \quad \lim_{k\to\infty} \frac{1}{n_k} (-\log |(f^n)'(x)|) = b_2$$

and $b_1/b_2 = \alpha$. Let τ be any weak*-limit of the sequence of measures

$$\tau_{n_k} := \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \delta_{f^j(x)},$$

where $\delta_{f^j(x)}$ is the Dirac measure supported at $f^j(x)$, compare Remark 2.1.14a. Then $\int \phi d\tau = b_1$ and $\int (-\log |f'|) d\tau = b_2$. Due to Choquet Theorem (Section 2 1) (or due to the Decomposition into Ergodic Components Theorem, Theorem 1.8.8) we can assume that τ is ergodic. Indeed, τ is an "average" of ergodic measures. So among all ergodic measure ν involved in the average, there is ν_1 such that $\frac{\int \phi d\nu_1}{\int -\log|f'|d\nu_1} \le \alpha$ and ν_2 such that $\frac{\int \phi d\nu_2}{\int -\log|f'|d\nu_2} \ge \alpha$. If $\alpha < -T'(\infty)$ we consider ν_1 as an ergodic τ, if $\alpha > -T'(-\infty)$ we consider ν_2. For the ergodic τ found in this way, the limit α can be different than for the original τ, but it will not belong to $[-T'(\infty), -T'(-\infty)]$ and we shall use the same symbol α to denote it. By Birkhoff's Ergodic Theorem applied to the functions ϕ and $\log|f'|$, for τ-a.e. x we have $\lim_{n\to\infty} \frac{S_n(\phi)(x)}{-\log|(f^n)'(x)|} = \alpha$. Hence, applying Lemma 7.8.3, we get

$$\alpha = d_{\mu_\phi}(x) = \frac{\int \phi d\tau}{-\int \log|f'|d\tau}.$$

Finally notice that by (7.8.9) there exists $q \in I\!\!R$ such that $\alpha = \frac{\int \phi d\mu_q}{-\int \log|f'|d\mu_q}$, whence $\alpha \in [-T'(\infty), -T'(-\infty)]$. This contradiction finishes the proof. ♣

Remark. We have proved in fact that for all $x \in X$ any limit number of the quotien $\log \mu_\phi(B(x,r)/\log r$ as $r \to 0$ lies in $[-T'(\infty), -T'(-\infty)]$, the fact stronger than $d_{\mu_\phi}(x) \in [-T'(\infty), -T'(-\infty)]$ for all x in the regular part of X.

6c. $F(-T'(\pm\infty)) = \mathbf{HD}\big(X_{-T'(\pm\infty)}\big)$**.** Consider any τ being a weak*-limit of a subsequence of μ_q as q tends to, say, ∞. We shall try to proceed with τ similarly as we did with μ_q, though we shall meet some difficulties. We do not know whether τ is ergodic (and choosing an ergodic one from the ergodic decomposition we may loose the convergence $\mu_q \to \tau$). Nevertheless using Birkhoff Ergodic Theorem and proceeding as in the proof of Lemma 7.8.3, we get

$$\begin{aligned}\frac{\int \lim_{n\to\infty} \frac{1}{n} S_n\phi(x)\, d\tau(x)}{-\int \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)|\, d\tau(x)} &= \frac{\int \phi\, d\tau}{-\int \log|f'|\, d\tau} = \lim_{q\to\infty} \frac{\int \phi d\mu_q}{-\int \log|f'| d\mu_q} \\ &= \lim_{q\to\infty} (-T'(q)) = -T'(\infty)\end{aligned}$$

with the convergence over a subsequence of q's. Since we know already that

$$d_{\mu_\phi}(x) = \frac{\lim_{n\to\infty} \frac{1}{n} S_n\phi(x)}{-\lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)|} \ge -T'(\infty),$$

we obtain for every x in a set $\tilde{X}_\tau$ of full measure τ that the limit $d_\tau(x) = -T'(\infty)$. We conclude with $\tilde{X}_\tau \subset X_{-T'(\infty)}$. Now we use the Volume Lemma for the measure τ. There is no reason for it to be Gibbs, neither ergodic, so we need to refer to the version of Volume Lemma coming from Theorem 7.1.10. We obtain

$$\begin{aligned}\mathrm{HD}(X_{-T'(\infty)}) \ge \mathrm{HD}^*(\tau) \ge \frac{\mathrm{h}_\tau(f)}{\int \log|f'|\, d\tau} &\ge \liminf_{q\to\infty} \frac{\mathrm{h}_{\mu_q}(f)}{\chi_{\mu_q}(f)} \\ &= \lim_{q\to\infty} T(q) - qT'(q) = F(-T'(\infty)).\end{aligned}$$

We have used here the upper semicontinuity of the entropy function $\nu \to \mathrm{h}_\nu(f)$ at τ due to the expanding property (see Theorem 2.4.6). It is only left to estimate $\mathrm{HD}(X_{-T'(\infty)})$ from above. As for μ_q, we have for every q and $x \in X_{-T'(\infty)}$ (see (7.8.6)) that

$$\lim_{r\to 0} \frac{\log \mu_q(B)}{\log r} = T(q) + q \lim_{r\to 0} \frac{\log \mu_\phi(B)}{\log r} = T(q) - qT'(\infty) \le T(q) - qT'(q),$$

Hence $\mathrm{HD}(X_{-T'(\infty)}) \le T(q) - qT'(q)$. Letting $q \to \infty$ we obtain $\mathrm{HD}(X_{-T'(\infty)}) \le F(-T'(\infty))$.

7. HP and Rényi spectra. Recall that topological supports of μ_ϕ and μ_q are equal to X, since these measures as Gibbs states for Hölder functions, do not vanish on open subsets of X due to Proposition 4.2.5. For every $\mathcal{G}_r$ a finite or countable covering X by balls of radius r of multiplicity at most C we have

$$1 \le \sum_{B\in\mathcal{G}_r} \mu_q(B) \le C.$$

Hence, by Lemma 7.8.2

$$C^{-1} \le r^{T(q)} \sum_{B\in\mathcal{G}_r} \mu_\phi(B)^q \le C$$

with an appropriate another constant C. Taking logarithms and, for $q \ne 1$, dividing by $(1-q)\log r$ yields (e) for $q \ne 1$.

8. Information dimension. For $q = 1$ we have $\lim_{q\to 1, q\ne 1} \frac{T(q)}{1-q} = -T'(1)$ by the definition of derivative. It is equal to $\mathrm{HD}(\mu_\phi) = \mathrm{PD}(\mu_\phi)$ by (a) and (b) and equal to $I(\mu_\phi)$ by Exercise 7.8.5. ♣

Exercises

7.2.1. Prove the equalities of Rényi and Hentschel-Procaccia spectra.

7.2.2. Prove Proposition 7.8.4 about Legendre transform pairs and remarks preceding and following it.

7.2.3. Prove for $\alpha = -T'(1)$ that $F(\alpha) = \alpha$ and $F'(\alpha) = 1$ (see Fig.1.) and $F'(-T'(\pm\infty)) = \pm\infty$.

7.2.4. Prove that if ϕ is not cohomologous to $-\mathrm{HD}(X)\log|f'|$ then the singular part $\hat{X}$ of X is nonempty. Moreover $\mathrm{HD}(\hat{X}) = \mathrm{HD}(X)$.

Hint: Using the Shadowing Lemma from Chapter 3, find trajectories that have blocks close to blocks of trajectories typical for $\mu_{-\mathrm{HD}(X)\log|f'|}$ of length N interchanging with blocks close to blocks typical for μ_ϕ of length εN, for N arbitrarily large and $\varepsilon > 0$ arbitrarily small.

7.2.5. Define the *lower* and *upper information dimension* $\underline{I}(\nu)$ and $\overline{I}(\nu)$ replacing in the definition of $I(\nu)$ the limit $\lim_r$ by the lower and upper limits respectively. Prove that $\mathrm{HD}_*(\nu) \le \underline{I}(\nu) \le \overline{I}(\nu) \le \mathrm{PD}^*(\nu)$, see (7.8.0).

Sketch of the proof. For an arbitrary $\varepsilon > 0$ there exist $C > 0$ and $A \subset X$, with $\nu(X \setminus A) \le \varepsilon$ such that for all r small enough there exists a partition $\mathcal{F}_r$ of A, satisfying

$H_\nu(r) + \varepsilon \ \geq \ -\sum_{B\in\mathcal{F}_r} \nu(B)\log\nu(B) \ \geq \ \sum_{B\in\mathcal{F}_r} \nu(B)\mathrm{HD}_*(\nu)\log\frac{1}{C\mathrm{diam}B} \ \geq \ \mathrm{HD}_*(\nu)(1-\varepsilon)\log\frac{1}{Cr}$.

On the other hand for the partition $\mathcal{B}_r$ of X into intersections with boxes (cubes) of sides of length r (compare Proposition 6.4.6 and the partition involved in the definition of Rényi dimension, but consider here disjoint cubes, that is open from one side), we have

$$\begin{aligned}\overline{I}(\nu) &= \limsup_{r\to 0}\frac{H_\nu(r)}{-\log r} \leq \limsup_{r\to 0}\frac{-\sum_{B\in\mathcal{B}_r}\nu(B)\log\nu(B)}{-\log r} \leq \limsup_{r\to 0}\frac{\int\log\nu(B_r(x))\,d\nu(x)}{\log r} \\ &\leq \int\Big(\limsup_{r\to 0}\frac{\log\nu(B_r(x))}{\log r}\Big)d\nu(x) \leq \mathrm{PD}^*(\nu),\end{aligned}$$

where $B_r(x)$ denotes the cube of side r containing x.
Prove that it has been eligible here to use cubes instead of balls standing in the definition of $\overline{d}_\nu(x)$. For this aim prove that for ν-a.e. $x\in X$, we have $\lim\frac{\log\nu(B_r(x))}{\log\nu(B(x,r))} = 1$. Use Borel-Cantelli lemma.
Prove that we could use Fatou's lemma (changing the order of limsup and integral) indeed due to the existence of a ν-integrable function which bounds from above all the functions $\log\nu(B(x,r))/\log r$ (or $\log\nu(B_r(x))/\log r$). Use again Borel-Cantelli lemma, for, say, $r = 2^{-k}$.

7.2.6. Let $\mu = \mu_\phi$ be a measure of maximal entropy on a topologically exact conformal expanding repeller X such that every point $x\in X$ has exactly d preimages (so $\phi = -\log d$). Prove (deduce from Theorem 7.8.5) that $F(\alpha) = \sup_{t\in I\!R}\Big(t + \frac{\alpha\mathrm{P}(t)}{\log d}\Big)$, more concretely $F(\alpha) = T + \frac{\alpha\mathrm{P}(T)}{\log d}$, where $\alpha = -\frac{\log d}{P'(T)}$.

7.2.7 Let $\phi_i : X \to I\!R$ be Hölder continuous functions for $i = 1,...,k$ and μ_{ϕ_i} their Gibbs measures. Define $X_{\alpha_1,\dots,\alpha_k} = \{x\in X : d_{\mu_i}(x) = \alpha_i \text{ for all } i = 1,...,k\}$. Define $\phi_{q_1,\dots,q_k,t} = -t\log|f'| + \sum_i q_i\phi_i$ and $T(q_1,...,q_k)$ as the only zero of the function $t\mapsto \mathrm{P}(\phi_{q_1,\dots,q_k,t})$. Prove the same properties of T as in Theorem 7.8.5, in particular that

$$\mathrm{HD}(X_{\alpha_1,\dots,\alpha_k}) = \inf_{(q_1,\dots,q_k)\in I\!R^k}\sum_i q_i\alpha_1 + T(q_1,...,q_k)$$

wherever the infimum is finite.

Historical and bibliographical notes

The section on multifractal analysis relies mainly on the monographs by Y. Pesin [P] and K. Falconer [F3] (though details are modified, for example we do not use Markov partition). The reader can find there comprehensive expositions and further references. The development of this theory has been stimulated by physicists, the paper often quoted is [HJKPS].

Bibliography

[F3] K. Falconer "Techniques in Fractal Geometry". John Wiley and Sons, Chichester, 1997.

[HJKPS] T.C. Hasley, M. Jensen, L. Kadanoff, I. Procaccia, B. Shraiman "Fractal measures and their singularities: the characterization of strange sets". Phys. Rev. A 33.2 (1986), 1141-1151.

[P] Y.B. Pesin "Dimension Theory in Dynamical Systems". The University of Chicago Press, Chicago and London, 1997.

Section 2. Fluctuations for Gibbs measures.

Theorem 7.2.1. Let $f : X \to X$ be a holomorphic expanding repeller. Let ϕ be a Hölder continuous function and let μ_ϕ be it's Gibbs measure. Then, with $\kappa = \mathrm{HD}(\mu_\phi)$, either

(a) $\kappa = \mathrm{HD}(X)$ and $\mu_\phi \asymp \mathrm{H}_\kappa$ on X (if $\psi = \phi + \kappa \log|f'| - \mathrm{P}(\phi)$ is a coboundary) or

(b) $\mu_\phi \perp \mathrm{H}_\kappa$ (if $\psi = \phi + \kappa \log|f'| - \mathrm{P}(\phi)$ is not a coboundary) and moreover, there exists $c_0 > 0$, ($c_0 = \sqrt{2\sigma^2(\psi)/\chi_{\mu_\phi}(f)}$), such that with the gauge function $\phi_c(r) = r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})$, where $\log_3$ means the iteration of the log function 3 times:

(c) $\mu_\phi \perp \mathrm{H}_{\phi_c}$ for all $0 < c < c_0$, and

(d) $\mu_\phi \ll \mathrm{H}_{\phi_c}$ for all $c > c_0$.

Proof. If ψ is a coboundary, then it follows from equality $\phi - \psi = -\kappa \log|f'| + \mathrm{P}(\phi)$ that $\mathrm{P}(\phi) = \mathrm{P}(\phi - \psi) = \mathrm{P}(\phi) + \mathrm{P}(-\kappa \log|f'|)$. Thus $\mathrm{P}(-\kappa \log|f'|) = 0$ and the part (a) follows immediately from Theorem 7.1.1 and the observation saying that the potential cohomological up to an additive constant have the same Gibbs states.

Suppose now that ψ is not a coboundary. As in the previous section let $\lambda = \inf|f'| > 1$. Recall that then there exists $\tau > 0$ small enough that firstly, $f\mid_A$ is one - to - one for all sets $A \subset X$ with $\mathrm{diam}(A) \le \tau$ and secondly, $d(f(y), f(z)) \ge \lambda d(y,z)$ if $d(y,z) < \tau$. Fix $x \in X$ and $r > 0$ such that $r < \tau$ Define $n = n(x,r)$ to be the least number such that $\mathrm{diam}(f^n(B(x,r))) \ge \tau$. Since $f : X \to X$ is topologically exact, $n(x,r)$ is finite. By definition of n, $\mathrm{diam}(f^j(B(x,r))) < \tau$ for all $j = 0, 1, ...n-1$. Therefore $f^n\mid_{(B(x,r))}$ is one - to - one and

$$m_\phi(f^n(B(x,r))) = \int_{B(x,r)} \exp(P(\phi)n - S_n\phi(z))dm_\phi(z),$$

where m_ϕ is the fixed point of the dual operator of $\mathcal{L}_\phi$, the transfer operator associated with the function $\phi - \mathrm{P}(\phi)$. Since the function ϕ is Hölder continuous, it follows from ??? that there exists a constant $K > 0$ such that $|S_n\phi(z) - S_n\phi(x)| \le K$ for all $x \in X$, $z \in B(x,r)$, and $n = n(x,r)$. Hence, we get

$$\begin{aligned} e^{-K}\exp(P(\phi)n - S_n\phi(x))m_\phi(B(x,r)) &\le m_\phi(f^n(B(x,r))) \\ &\le e^K \exp(P(\phi)n - S_n\phi(x))m_\phi(B(x,r)) \end{aligned}$$

or equivalently

$$\begin{aligned} e^{-K}\exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r))) &\le m_\phi(B(x,r)) \\ &\le e^K \exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r))). \end{aligned}$$

Similarly, since $\log|f'(z)|$ is a Hölder continuous function, there exists a constant $K_1 > 0$ such that

$$|S_n(\log|f'(z)|) - S_n(\log|f'(x)|)| \le K_1$$

for all $x \in X$, $z \in B(x,r)$, and $n = n(x,r)$. Then

$$\frac{|(f^n)'(z)|}{|(f^n)'(x)|} \le e^{K_1}$$

Therefore

$$\begin{aligned}\operatorname{diam}(B(x,r))e^{-K_1}|(f^n)'(x)| &\le \operatorname{diam}(f^n(B(x,r)))\\ &\le \operatorname{diam}(B(x,r))|(f^n)'(x)|e^{K_1}.\end{aligned}$$

Hence, $\operatorname{diam}(B(x,r))e^{-K_1}|(f^n)'(x)| \le \tau\|f'\|$, or equivalently $r \le \frac{1}{2}e^{K_1}|(f^n)'(x)|^{-1}\tau\|f'\|$. Also, since $\operatorname{diam}(f^n(B(x,r))) \ge \tau$ we get similarly $r \ge \frac{1}{2}e^{-K_1}|(f^n)'(x)|^{-1}\tau$. In conclusion,

$$\frac{1}{2}e^{-K_1}|(f^n)'(x)|^{-1}\tau \le r \le \frac{1}{2}e^{K_1}|(f^n)'(x)|^{-1}\tau\|f'\|. \tag{7.2.1}$$

Hence, denoting $\sqrt{\log(|(f^n)'(x)|)\log_3(|(f^n)'(x)|)}$ by $g_n(x)$, we get

$$\begin{aligned}\frac{m_\phi(B(x,r))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})} &\le \frac{e^K\exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r)))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})}\\ &\le \frac{e^K\exp(S_n\phi(x) - P(\phi)n)}{(\frac{1}{2}\frac{1}{|(f^n)'(x)|e^{K_1}}\tau)^\kappa\exp(c\sqrt{\log(2e^{-K_1}\frac{|(f^n)'(x)|}{\tau\|f'\|})\log_3(2e^{-K_1}\frac{|(f^n)'(x)|}{\tau\|f'\|})})}\\ &\le \frac{Q\exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))},\end{aligned}$$

where Q is a large enough constant. Similarly,

$$\begin{aligned}&\frac{m_\phi(B(x,r))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})}\\ &\ge \frac{e^{-K}\exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r)))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})}\\ &\ge \frac{e^{-K}\exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r)))}{(\frac{1}{2}|(f^n)'(x)|^{-1}e^{K_1}\tau\|f'\|)^\kappa\exp(c\sqrt{\log(2e^{K_1}|(f^n)'(x)|\tau^{-1})\log_3(2e^{K_1}|(f^n)'(x)|\tau^{-1})})}\\ &\ge \frac{Q_1\exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r)))}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))} \quad (Q_1 \text{ large enough})\\ &\ge \frac{Q_2\exp(S_n\phi(x) - p(\phi)n)}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))},\end{aligned}$$

where $Q_2 = Q_1\min\{m_\phi(f^n(B(x,r))) : x \in X, r > 0\}$. Note that, since the topological support of μ_ϕ and m_ϕ is equal to X, using the the Bounded Distortion Theorem we get $Q_2 > 0$ because $f^n(B(x,r)) \supset B(f^n(x), r|(f^n)'(x)|K^{-1}) \supset B(f^n(x), R)$, where $R = K^{-1}\tau e^{-K_1}$. Finally, we get

$$0 < \frac{Q_2 \exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa} \exp(cg_n(x))} \leq \frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})} \leq \frac{Q \exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa} \exp(cg_n(x))}$$

Hence,

$$\begin{aligned} \left[\frac{Q_2 \exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa} \exp(cg_n(x))}\right] &\leq \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right]. \\ &\leq \log\left[\frac{Q \exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa} \exp(cg_n(x))}\right] \end{aligned}$$

Therefore,

$$\begin{aligned} &\log Q_2 + S_n\phi(x) - P(\phi)n + \kappa \log|(f^n)'(x)| - cg_n(x) \\ &\quad\leq \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] \\ &\quad\leq \log Q + S_n\phi(x) - P(\phi)n + \kappa \log|(f^n)'(x)| - cg_n(x), \end{aligned}$$

or equivalently,

$$\begin{aligned} &\log Q_2 + g_n(x)\left[\frac{S_n\phi(x) - P(\phi)n + \kappa\log|(f^n)'(x)|}{g_n(x)} - c\right] \\ &\quad\leq \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] \\ &\quad\leq \log Q + g_n(x)\left[\frac{S_n\phi(x) - P(\phi)n + \kappa\log|(f^n)'(x)|}{g_n(x)} - c\right]. \end{aligned} \tag{7.2.1}$$

So, in order to proceed further, we are tempted to evaluate the following upper limit.

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{\log|(f^n)'(x)| \log_3|(f^n)'(x)|}}$$

First, by the Birkhoff ergodic theorem, for every $\epsilon > 0$ there exists $X_1 \subset X$ such that $m_\phi(X_1) = 1$ and forall $x \in X_1$ there exists $N > 0$ such that $\forall n \geq N$

$$n(\chi_\mu - \epsilon) \leq \log|(f^n)'(x)| \leq n(\chi_\mu + \epsilon).$$

Second, since ψ is not a coboundary, by the Law of Iterated Logarithms (see ???), there exists $X_2 \subset X$ such that $m_\phi(X_2) = 1$, and forall $x \in X_2$

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{n\log_2(n)}} = \sqrt{2\sigma^2}.$$

Therefore, for all $x \in X_1 \bigcap X_2$

$$\begin{aligned}
\limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} &\le \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{n(\chi_\mu-\epsilon)\log_2(n(\chi_\mu-\epsilon))}} \\
&\le \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{(\chi_\mu-2\epsilon)}\sqrt{n\log_2(n))}} \\
&= \frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu-2\epsilon)}}.
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} &\ge \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{n(\chi_\mu+\epsilon)\log_2(n(\chi_\mu+\epsilon))}} \\
&\ge \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{(\chi_\mu+2\epsilon)}\sqrt{n\log_2(n))}} \\
&= \frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu+2\epsilon)}}.
\end{aligned}$$

In conclusion, forall $x \in X_1 \bigcap X_2$, we have

$$\frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu+2\epsilon)}} \le \limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} \le \frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu-2\epsilon)}}$$

Hence, letting $\epsilon \searrow 0$,

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} = \sqrt{\frac{2\sigma^2}{\chi_\mu}}.$$

Thus, with

$$\beta_n(c) = g_n(x)\left[\frac{S_n\phi(x) - P(\phi)n + \kappa\log|(f^n)'(x)|}{g_n(x)} - c\right],$$

we obtain

$$\limsup_{n\to\infty} \beta_n(c) = \begin{cases} \infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} > c \\ -\infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} < c \end{cases} \tag{7.2.3}$$

since, $\lim_{n\to\infty} g_n(x) = \infty$, and this is because f is expanding. By (7.2.1)

$$\begin{aligned}
\limsup_{n\to\infty}[\log Q_2 + \beta_n(c)] &\le \limsup_{r\to 0} \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] \\
&\le \limsup_{n\to\infty}[\log Q + \beta_n(c)].
\end{aligned}$$

By (7.2.3), it gives

$$\limsup_{r\to 0} \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] = \begin{cases} \infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} > c \\ -\infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} < c \end{cases}.$$

In other words,

$$\limsup_{r\to 0} \frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})} = \begin{cases} \infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} > c \\ 0 & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} < c \end{cases}.$$

Therefore, by Theorem 5.??, $\mu_\phi \perp \mathrm{H}_{\phi_c}$ for all $c < \sqrt{\frac{2\sigma^2}{\chi_\mu}}$ and $\mu_\phi \ll \mathrm{H}_{g_c}$ for all $c > \sqrt{\frac{2\sigma^2}{\chi_\mu}}$. The proof is finished. ♣

Note that this proof is done without the use of Markov partitions. Note also that the last display in the proof of Theorem 7.2.1 is known as a refined volume lemma.

Section 7.3. Radial behaviour of the Riemann map,I.

In this section $f : X \to X$ continues to be a conformal expanding repeller and we assume additionally that X is a Jordan curve. We then call f a boundary expending conformal repeller. Let $\Omega \subset \overline{\mathbb{C}}$ be one of the components of $\overline{\mathbb{C}} \setminus X$ and fix $z_0 \in \Omega$. Let $D^1 = \{z : |z| < 1\}$. In view of Caratheodory's theorem, any Riemann map $R : D^1 \to \Omega$ (conformal homeomorphism) sending zero to z_0 (which is unique up to rotation) extends homeomorphically to $\overline{D^1}$. For more information about a Riemann map we refer the reader to [?]. We also assume that there exists an open topological annulus $A \subset \overline{\mathbb{C}}$ surronding $\partial\Omega$ such that $f(A \cap \Omega) \subset \Omega$. Since f is a local homeomorphism, it is easy to see that $f^{-1}(A \cap \Omega) \subset \Omega$. With all these assumptions we speak about the expanding map $f : \partial\Omega \to \partial\Omega$ as a conformal boundary repeller with the Jordan domain Ω. This enables us to lift the map f to the closed topological annulus $B = R^{-1}(A \cap \Omega) \subset \overline{D^1}$ one of whose boundary components is the circle S^1 by setting

$$g = R^{-1} \circ f \circ R : \overline{B} \to \overline{D^1}.$$

Obviously $g(B) \subset \overline{D^1}$. Denoting by $I : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the inversion with respect to the unit circle S^1 and applying the Schwarz reflection principle we see see that g extends analytically to the topological annulus $F = B \cup I(B)$. Our first aim is to show that g is a boundary expending conformal repeller. We begin with the following.

Lemma 7.3.1. There exists $\delta > 0$ such that for every $z \in S^1$ there exists a unique holomorphic inverse branch $g_z^{-n} : B(g^n(z), 2\delta) \to F$ sending $g^n(z)$ to z. If n is large enough (independent of z), then $|(g_z^{-n})'(w)| \le 1/2$ for all $w \in B(g^n(z), \delta)$. In particular the map $g : S^1 \to S^1$ is expanding.

Proof. Fix $\delta > 0$ so small that $B(S^1, 2\delta) \subset F$. Fix $z \in S^1$ and consider the ball $B(g^n(z), 2\delta)$. Define the inverse branch $g_z^{-n} : \overline{D^1} \cap B(g^n(z), 2\delta) \to \mathbb{C}$ of g^n by putting $g_z^{-n}(w) = R^{-1} \circ f_{R(z)}^{-n} \circ R(w)$, where $f_{R(z)}^{-n}$ is a local holomorphic inverse branch of f^n defined throughout a neighbourhood of $f^n(R(z))$ sending $f^n(R(z))$ to $R(z)$. Notice that $g_z^{-n}(g^n(z)) = z$, g_z^{-n} is continuous and restricted to the set $D^1 \cap B(g^n(z), 2\delta)$ is analytic. Since $f^{-n}(A \cap \overline{\Omega}) \subset A \cap \Omega$, we conclude that $g_z^{-n}(\overline{D^1} \cap B(g^n(z), 2\delta) \subset F \cap \overline{D^1}$. Since in addition $I(B(g^n(z), 2\delta)) = B(g^n(z), 2\delta)$ and $I(F) = F$, applying the Schwarz reflection principle again, we conclude that g_z^{-n} extends to an analytic map $g_z^{-n} : B(g^n(z), 2\delta) \to F$. Since $g^n \circ g_z^{-n}$ is an identity map on $D^1 \cap B(g^n(z), 2\delta)$ and g_z^{-n} is analytic on this intersection, we conclude that $g^n \circ g_z^{-n}$ is an identity map on the entire ball $B(g^n(z), 2\delta)$. This means that $g_z^{-n} : B(g^n(z), 2\delta) \to F$ is a holomorphic inverse branch of g^n sending $g^n(z)$ to z. Since

$$\lim_{n\to\infty} \sup_{z\in S^1} \left\{ \mathrm{diam} \left(f_{R(z)}^{-n} \big(R(B(g^n(z), 2\delta)) \big) \right) \right\} = 0,$$

since

$$g_z^{-n}\big(\overline{D^1} \cap B(g^n(z), 2\delta)\big) = R^{-1} \left(f_{R(z)}^{-n} \big(R(B(g^n(z), 2\delta)) \big) \right)$$

and since $R^{-1} : \overline{D^1} \to \overline{\Omega}$ is a uniformly continuous funtion, we get

$$\lim_{n\to\infty} \sup_{z\in S^1} \left\{ \mathrm{diam} \left(g_z^{-n}(\overline{D^1} \cap B(g^n(z), 2\delta) \right) \right\} = 0.$$

Since

$$g_z^{-n}\big(B(g^n(z), 2\delta) \cap (\mathbb{C} \setminus D^1)\big) = I\big(g_z^{-n}\big(B(g^n(z), 2\delta) \cap (\mathbb{C} \setminus D^1)\big)\big),$$

we thefore see that

$$\lim_{n\to\infty} \sup_{z\in S^1} \left\{ \mathrm{diam} \left(g_z^{-n}(B(g^n(z), 2\delta) \right) \right\} = 0.$$

Hence, applying Koebe's distortion theorem, we get that

$$\lim_{n\to\infty} \sup_{z\in S^1} \{ |(g_z^{-n})'(w)| : w \in B(g^n(z), \delta) \} = 0.$$

There thus exists $n \geq 1$ such that

$$\sup_{z\in S^1} \{ |(g_z^{-n})'(w)| : w \in B(g^n(z), \delta) \} \leq \frac{1}{2}.$$

Hence, for every $z \in S^1$, $|(g^n)'(z)| = |(g_z^{-n})'(g^n(z))|^{-1} \geq 2$ and the proof is complete. ♣

Lemma 7.3.2. There exists $\delta > 0$ such that if $\{x, g(x), g^2(x), \dots, g^n(x)\} \subset B(S^1, \delta)$, then there exists a unique holomorphic inverse branch g_x^{-n} of g^n defined on $B(g^n(x), 4\delta)$ and sending $g^n(x)$ to x. In addition, there exist $\lambda > 1$ and $C > 0$ such that $|(g_x^{-n})'(w)| \leq C\lambda^{-n}$ for all $w \in B(g^n(x), 4\delta)$.

Proof. It follows from Lemma 7.3.1 that there exist $c > 0$ and $\eta > 1$ such that for every $z \in S^1$ and all $w \in B(g^n(z), \delta)$,

$$(7.3.1) \qquad |(g_z^{-n})'(w)| \le C\eta^{-n}.$$

Fix $q \ge 1$ so large that $C\eta^{-q} \le 1/6$. There obviously exists $0 < \theta < \delta/6$ such that for every $1 \le j \le q$ and every $x \in \mathbb{C}$ such that $\{x, g(x), g^2(x), \dots, g^j(x)\} \subset B(S^1, \theta)$, there exists a unique holomorphic branch $g_x^{-j} : B(g^j(x), 4\theta) \to F$ of g^j sending $g^j(x)$ to x. Suppose now that $j = q$. Since $B(g^q(x), 2\theta) \cap S^1 \ne \emptyset$, we can take a point z lying in this intersection. Since $g^{-q}(S^1) = S^1$, we have $g_x^{-q}(z) \in S^1$. Since in addition $B(g^q(x), 4\theta) \subset B(z, 6\theta) \subset B(z, \delta)$, it follows (7.3.1) that

$$(7.3.2) \qquad |(g_x^{-q})'(w)| \le \frac{1}{6}$$

for all $w \in B(g^q(x), 4\theta)$. Recalling the choice of w we therefore obtain that

$$(7.3.3) \qquad g_x^{-q}(B(g^q(x), 4\theta)) \subset B(S^1, \theta) \cap B(x, \theta)$$

Thus given a piece $\{x, g(x), \dots, g^n(x)\}$ of the forward trajectory of x, we can split it into blocks

$$g^n(x), g^{n-1}(x), \dots, g^{n-q}(x);\ g^{n-q}(x), g^{n-q-1}(x), \dots, g^{n-2q}(x);\ \dots, x$$

all of them of length q except the last one of length $i \le q$. In view of (7.3.3) we can inductively form the composition

$$g_x^{-i} \circ \dots \circ g_{g^{n-3q}(x)}^{-q} \circ g_{g^{n-2q}(x)}^{-q} \circ g_{g^{n-q}(x)}^{-q}$$

which is an inverse branch g_x^{-n} of g^n defined on $B(g^n(x), 4\theta)$ and sending $g^n(x)$ to x. Writing $n = pq + i$, $0 \le i \le q - 1$ we see from (7.3.2) that for every $w \in B(g^n(x), 4\theta)$, we have

$$|(g_x^{-n})'(w)| \le M \left(\frac{1}{6}\right)^p \le M \left(\frac{1}{6}\right)^{\frac{n}{q}-1},$$

where $M = (\min_{j \le q-1} \inf_{z \in F}\{|(g^j)'(z)|\})^{-1}$. We are done redefining δ to be θ. ♣

Proposition 7.3.3 With $U = B(S^1\delta)$, where δ comes from Lemma 7.3.2,the map $F : S^1 \to S^1$ is a boundary expanding repeller.

Proof. Since $R|_{S^1}$ establishes a topological conjugacy between $g|_{S^1}$ and $f|_{\partial\Omega}$, the map $g : S^1 \to S^1$ is topologically transitive. Since, by Lemma 7.3.1, $g : S^1 \to S^1$ is expanding, we only need to check that there exists an open set $U \subset F$ containing S^1 such that $\bigcap_{n \ge 0} g^{-n}(U) = S^1$. And indeed, suppose that $\{g^n(x) : n \ge 0\} \subset B(S^1, \delta)$, where δ is taken from Lemma 7.3.2. It follows from this lemma that for every $n \ge 0$ there exists a unique holomorphic inverse branch g_x^{-n} of g^n defined on $B(g^n(x), 4\delta)$ and sending $g^n(x)$ to x. In addition

$$\operatorname{diam}\big(g_x^{-n}(B(g^n(x), 3\delta))\big) \le 6\delta c\lambda^{-n}.$$

Since $B(g^n(x),3\delta)\cap S^1\neq\emptyset$, we can choose a point w in this intersection. Then $g_x^{-n}(w)\in S^1$ and therefore

$$\operatorname{dist}(x,S^1)\le|g_x^{-n}(g^n(x))-g_x^{-n}(w)|\le\operatorname{diam}\big(g_x^{-n}(B(g^n(x),3\delta)\big)\le 6\delta c\lambda^{-n}.$$

Letting $n\to\infty$, we therefore conclude that $x\in S^1$. Thus g becomes a boundary expanding repeller with $U=B(S^1,\delta)$. ♣

Our next result is the following the following.

Theorem 7.3.2. If μ is a g-invariant ergodic probability measure of of positive entropy on S^1, then the non-tangent limit

$$\lim_{x\to z}\frac{-\log|R'(x)|}{\log(1-|x|)}$$

exists for μ-almost every point $z\in S^1$, is constant almost everywhere (denote it by $\chi_\mu(R)$), and

$$\chi_\mu(R)=1-\frac{\chi_{\mu\circ R^{-1}}(f)}{\chi_\mu(g)}.$$

Proof. Given $z\in S^1$, $0<\alpha<\pi/2$, and $0<r<1$ let

$$S_{\alpha,r}(z)=z\cdot(1+\{x\in\mathbb{C}\setminus\{0\}:0<|x|\le r,\pi-\alpha\le\operatorname{Arg}(x)\le\pi+\alpha\}).$$

A straightforward trigonometrical argment shoows that for all $0<\alpha<\pi/2$, and $0<r<1$ there exists $0<\kappa<1/2$ such that for all $z\in S^1$ and all $x\in S_{\alpha,r}(z)$ we have

$$\kappa|x-z|\le 1-|x|. \tag{7.3.4}$$

Fix now $z\in S^1$ and $x\in S_{\alpha,r}(z)$. It follows from Proposition 7.3.3 that there exists $k\ge1$ such that $g^j(x)\in B(S^1,\delta)$ for all $0\le j\le k-1$ and $g^k(x)\notin B(S^1,\delta)$ which means that $1-|g^k(x)|\ge\delta$. Therefore there exists the least $n=n(x,z)\ge0$ such that

$$1-|g^{n+1}(x)|\ge\delta(8K\kappa^{-1})^{-1}, \tag{7.3.5}$$

where $K\ge1$ is the Koebe's constant assiciated with the scale 1/2. Hence

$$1-|g^j(x)|<\delta(4K\kappa^{-1})^{-1}<\delta \tag{7.3.6}$$

for all $0\le j\le n$. It therefore follows from Lemma 7.3.2 that thaere exists there a unique holomorphic inverse branch g_x^{-n} of g^n defined on $B(g^n(x),4\delta)$ and sending $g^n(x)$ to x. Let $w=g^n(x)/|g^n(x)|$. Then $w\in\overline{B}(g^n(x),1-|g^n(x)|)\cap S^1$ and using (7.3.4) along with Koebe's Distortion Theorem, we get

$$\begin{aligned}\kappa|x-z|&\le 1-|x|\\&\le|x-g_x^{-n}(w)|=|g_x^{-n}(g^n(x))-g_x^{-n}(w)|\le K|g^n(x)-g_x^{-n}(w)||(g^n)'(x)|^{-1}\\&=K(1-|g^n(x)|)|(g^n)'(x)|^{-1}.\end{aligned}$$

Hence, applying Koebe's $\frac{1}{4}$-Theorem, we obtain

$$
\begin{aligned}
&g_x^{-n}\big(B(g^n(x), 8K\kappa^{-1}(1-|g^n(x)|)\big) \supset B\big(x, 2K\kappa^{-1}(1-|g^n(x)|)|(g^n)'(x)|^{-1}\big)\\
&\qquad \supset B(x, 2|z-x|) \ni z. \qquad (7.3.7)
\end{aligned}
$$

It therefore follows from Koebe's Distortion Theorem that

$$
K^{-1} \le \frac{|(g^n)'(x)|}{|(g^n)'(z)|} \le K, \tag{7.3.8}
$$

that (using (7.3.4))

$$
\begin{aligned}
1-|g^n(x)| &\le |g^n(z)-g^n(x)| \le K|(g^n)'(x)|\cdot|z-x| \le K|(g^n)'(x)|\kappa^{-1}(1-|x|)\\
&= K\kappa^{-1}|(g^n)'(x)|(1-|x|), \qquad (7.3.9)
\end{aligned}
$$

and (using (7.3.7)) that

$$
\begin{aligned}
1-|x| &\le |z-x| = |g_x^{-n}(g^n(z)) - g_x^{-n}(g^n(x))| \le K|(g^n)'(x)|^{-1}|g^n(z)-g^n(x)|\\
&\le K|(g^n)'(x)|^{-1}8K\kappa^{-1}(1-|g^n(x)|) = 8K^2\kappa^{-1}(1-|g^n(x)|)|(g^n)'(x)|^{-1}. \qquad (7.3.10)
\end{aligned}
$$

Since the Riemann map : $D^1 \to \Omega$ is uniformly continuous, $R(g^n(x))$ lies close to $R(g^n(z))$. Let $f_{R(z)}^{-n}$ be a holomorphic inverse branch of f^n defined on some small neighbourhood of $R(g^n(z))$, containing $R(g^n(x))$ and sending $R(g^n(z)) = f^n(R(z))$ to $R(z)$. Then $f_{R(z)}^{-n}(R(g^n(x)) = x$ and applying Koebe's Distortion Theorem, we obtain

$$
\hat{K}^{-1} \le \frac{|(f^n)'(R(x))|}{|(f^n)'(R(z))|} \le \hat{K}, \tag{7.3.11}
$$

for some constant $\hat{K}$ independent of z, x and n. Since

$$
1-|g^{n+1}(x)| \le ||g'||(1-|g^n(x)|), \tag{7.3.12}
$$

Combining (7.3.5), (7.3.6), (7.3.9), (7.3.10) and (7.3.12), we get

$$
(8K||g'||)^{-1}\kappa^2\delta \le (1-|x|)|(g^n)'(x)| \le K\delta. \tag{7.3.13}
$$

By Birkhoff's Ergodic Theorem there exists a Borel set $Y \in S^1$ such that $\mu(Y)=1$ and

$$
\lim_{k\to\infty}\frac{1}{k}\log|(g^k)'(z)| = \chi_\mu(g) \quad\text{and}\quad \lim_{n\to\infty}\frac{1}{k}\log|(f^k)'(R(z))| = \chi_{\mu\circ R^{-1}}(f).
$$

for all $z \in Y$. Suppose that $z \in Y$. Fix also $\varepsilon > 0$. Then for all k sufficiently large

$$
\exp(k(\chi_\mu(g)-\varepsilon)) \le |(g^k)'(z)| \le \exp(k((\chi_\mu(g)+\varepsilon)), \tag{7.3.14}
$$

and

$$\exp(k(\chi_{\mu\circ R^{-1}}(f)-\varepsilon)) \le |(f^k)'(R(z))| \le \exp(k((\chi_{\mu\circ R^{-1}}(f)+\varepsilon)).$$

Combining these two formulas along with (7.3.8) and (7.3.11), we get for all n large enough (which can be assured by taking x sufficiently close to z),

$$\exp(n(\chi_\mu(g)-2\varepsilon)) \le |(g^n)'(x)| \le \exp(n(\chi_\mu(g)+2\varepsilon)), \tag{7.3.15}$$

and

$$\exp(n(\chi_{\mu\circ R^{-1}}(f)-2\varepsilon)) \le |(f^n)'(R(x))| \le \exp(n(\chi_{\mu\circ R^{-1}}(f)+2\varepsilon)) \tag{7.3.16.}$$

Combining (7.3.13) and (7.3.14), we get

$$\log(4||g'||)^{-1}\delta) - n(\chi_\mu(g)+2\varepsilon) \le \log(1-|x|) \le \log(K\delta) - n(\chi_\mu(g)-2\varepsilon).$$

Thus, for all n large enough

$$\frac{-\log(1-|x|)}{\chi_\mu(g)+3\varepsilon} \le n \le \frac{-\log(1-|x|)}{\chi_\mu(g)-3\varepsilon}. \tag{7.3.17}$$

In view of Koebe's Distortion Theorem, (7.3.12) and (7.3.5) there exists a constant C depending only on K, κ and $||g'||$ such that

$$C^{-1}|R'(0)| \le |R'(g^n(x))| \le C|R'(0)|$$

Therefore, differentiating the equality $R\circ g^n = f^n\circ R$, we get

$$|R'(x)| = |(g^n)'(x)|\frac{|R'(g^(x))|}{|(f^n)'(R(x))|} \in \left[C^{-1}|R'(0)|\frac{|(g^n)'(x)|}{|(f^n)'(R(x))|}, C|R'(0)|\frac{|(g^n)'(x)|}{|(f^n)'(R(x))|}\right].$$

Hence, using (7.3.15), (7.3.16) and (7.3.17)

$$\begin{aligned}|R'(x)| &\le C|R'(0)||(g^n)'(x)|\cdot|(f^n)'(R(x))|^{-1}\\ &\le C|R'(0)|\exp(n(\chi_\mu(g)+2\varepsilon))\exp(-n(\chi_{\mu\circ R^{-1}}(f)-2\varepsilon))\\ &\le \exp\left(\frac{-\log(1-|x|)}{\chi_\mu(g)-3\varepsilon}(\chi_\mu(g)-\chi_{\mu\circ R^{-1}}(f)+4\varepsilon)\right).\end{aligned}$$

Thus

$$\frac{-\log|R'(x)|}{\log(1-|x|)} \preceq \frac{\chi_\mu(g)-\chi_{\mu\circ R^{-1}}(f)+4\varepsilon}{\chi_\mu(g)-3\varepsilon}$$

and we obtain similarly

$$\frac{-\log|R'(x)|}{\log(1-|x|} \succeq \frac{\chi_\mu(g)-\chi_{\mu\circ R^{-1}}(f)-4\varepsilon}{\chi_\mu(g)+3\varepsilon}$$

So, letting $\varepsilon \to 0$ (which forces us to let $n(x, z) \to \infty$ and which in turn forces us to let $x \to z$) finishes the proof. ♣

In order to make use of this result we shall provide a simple proof of the following Proposition.

Section 7.4. Harmonic measure.

In this section we keep the notation of the previous one. The measure $l \circ R^{-1}$, the image of the image of the Lebesgue measure l on S^1 under the Riemann map is said to be the harmonic measure of $\partial\Omega$ with respect to (or viewed from) the point z_0. Since all the Riemann maps differ by compositions with M̈bius transformations preserving t he unit circle, all the harmonic measures are strongly equivalent and corresponding Radon-Nikodyn derivatives are bounded away from zero and infinity. In particular all the harmonic measures induce the same strong measure class, which as long as we are only interested in metric properties of this class, enables us to speak generally about a harmonic measure without specifying the point z_0. Writing ω we will actually mean the class of all measures equivalent with the harmonic measure with Radon-Nikodym derivatives bounded uniformly from above and below. For more information about harmonic measure we refer the reader to [?]. Our first aim is to represent harmonic measure as a Gibbs measure and then to apply the results of the previous section. Since the Hausdorff dimension of the circle S^1 is equal to 1, it follows from Theorem 7.1.1 that $P(g, \phi) = 0$, where $\phi = -\log|g'|$. Of course the Lebesgue measure l on S^1 is equivalent with H_1. Since R is a topological conjugacy between g and f on S^1, $P(f, \phi \circ R^{-1}) = P(g, \phi) = 0$. Since moreover R^{-1} is a metric conjugacy between metric dynamical systems $(f, \mu_\phi \circ R^{-1})$ and (g, μ_ϕ), we therefore have

$$\mathrm{h}_{\mu_\phi \circ R^{-1}}(f) + \int \phi \circ R^{-1}\, d\mu_\phi \circ R^{-1} = \mathrm{h}_{\mu_\phi}(g) + \int \phi\, d\mu_\phi = 0.$$

Since in addition $\mu_\phi \circ R^{-1} \in \omega$, the uniqueness of an equilibrium state for $\phi \circ R^{-1}$ results in the following.

Theorem 7.4.2. The harmonic measure ω coincides with the class of the Gibbs states of the map $f : X \to X$ and the Hölder continuous potential $-\log|g'| \circ R^{-1}$.

We now want to argue that $\mathrm{HD}(\omega) = 1$. This is a general result due to Makarov (see ???) true for any simply connected domain Ω with no dynamics involved. We shall however provide here a proof in the dynamical context only which is shorter and simpler than the general one.

Proposition 7.4.3. If ω is a harmonic measure on the boundary of a Jordan domain Ω, then $\chi_l(R)(z) = 0$ for l-a.e. $z \in S^1$, where l denotes the normalized Lebesgue measure on S^1.

Proof. Fix $z \in S^1$ and $0 < r < 1$. Then by the Koebe distortion theorem $2^{-1}(1-r)^{-3} \leq |R'(z)| \leq (1-r)^{-3}$ which implies that

$$2 \leq \frac{-\log|R'(rz)|}{\log(1-r)} \leq 4$$

for all r sufficiently close to 1. Since the Gibbs state of the function $-\log|g'|$ is a g-invariant probability measure absolutely continuous with respect to the Lebesgue measure it follows from Theorem 7.4.2 that $\chi_l(R)(z)$ is constant for l-a.e. $z \in S^1$. Hence

$$\begin{aligned}\chi_l(R) &= \int \chi_l(R)\, dl = \int \lim_{r\to 1} \frac{\log|R'(rz)|}{-\log(1-r)}\, dl(z) \\ &= \lim_{r\to 1} \frac{1}{-\log(1-r)} \int \log|R'(rz)|\, dl(z).\end{aligned}$$

Since $\log|R'|$ is a harmonic function we can continue the above chain of equalities writing

$$\chi_l(R) = \lim_{r\to 1} \frac{r\log|R'(0)|}{-\log(1-r)} = 0.$$

The proof is finished. ♣

Corollary 7.4.4. Suppose that an expanding map $f : \partial\Omega \to \partial\Omega$ is a conformal boundary repeller, and Ω is a Jordan domain. Then $\mathrm{HD}(\omega) = 1$, for ω, the harmonic measure viewed from Ω.

proof. Let μ be the Gibbs state on S^1 corresponding to the potential $-\log|g'|$. Simultaneously μ is the unique probability g-invariant measure equivalent with the Lebesgue measure l on S^1. Then $\mu \circ R^{-1}$ is a probability f-invariant measure equivalent to ω. In view of Theorem 7.4.2 and Lemma 7.4.3, $\chi_{\mu\circ f^{-1}}(f) = \chi_\mu(g) > 0$. Since $R : S^1 \to \partial\Omega$ is a topological conjugacy between $g : S^1 \to S^1$ and $f : \partial\Omega \to \partial\Omega$, $\mathrm{h}_{\mu\circ f^{-1}}(f) = \mathrm{h}_\mu(g)$. Thus

$$\frac{\mathrm{h}_{\mu\circ f^{-1}}(f)}{\chi_{\mu\circ f^{-1}(f)}} = \frac{\mathrm{h}_\mu(g)}{\chi_\mu(g)}.$$

Since $\mathrm{HD}(\mu) = 1$, an immediate application of Lemma 7.1.4 (Volume Lemma) finishes the proof. ♣

Theorem 7.4.5. Let $f : \partial\Omega \to \partial\Omega$ be a conformal boundary repeller, where Ω is a Jordan domain. Then either

(a) $\omega \asymp \mathrm{H}_1$ on $\partial\Omega$ (if $\log|g'|$ and $\log|f' \circ R|$ are cohomologous) or

(b) $\omega \perp \mathrm{H}_1$ (if $\log|g'|$ and $\log|f' \circ R|$ are not cohomologous), and then there exists $c_0 > 0$ such that with the gauge function $\phi_c(t) = t\exp(c\sqrt{\log(1/t)\log_3(1/t)})$,

$$\omega \perp \mathrm{H}_{\phi_c} \quad \text{for all} \quad 0 \leq c < c_0$$

and

$$\omega \ll \mathrm{H}_{\phi_c} \quad \text{for all } c > c_0.$$

Proof. If $\log|g'|$ and $\log|f' \circ R|$ are cohomologous, then also the functions $-\log|g' \circ R^{-1}|$ and $-\log|f'|$ are cohomologous (with respect to the map $f : \partial\Omega \to \partial\Omega$). By Theorem 7.4.2 ω is a Gibbs state of the potential $-\log|g' \circ R^{-1}|$. Since $\mathrm{P}(f, -\log|f'|) = \mathrm{P}(f, -\log|g' \circ R^{-1}|) = \mathrm{P}(g, -\log|g'|) = 0$, it follows from Theorem 7.1.1 that $\mathrm{HD}(\partial\Omega) = 1$, and the Gibbs states of the potential $-\log|f'|$ are equivalent to the 1-dimensional Hausdorff measure on $\partial\Omega$. So, the part (a) of Theorem 7.4.5 is proved.
Suppose now that $\log|g'|$ and $\log|f' \circ R|$ are not cohomologous. Then $-\log|g' \circ R^{-1}|$ and $-\log|f'|$ are not cohomologous. Let μ be the invariant Gibbs state of $-\log|g' \circ R^{-1}|$. By Corollary 7.4.4 and Theorem 7.a.2, $\mathrm{HD}(\mu) = 1$. Hence $\psi = -\log|g' \circ R^{-1}| + \mathrm{HD}(\mu)\log|f'| - \mathrm{P}(f, -\log|g' \circ R^{-1}|)$ is not a coboundary and, since by Theorem 7.4.2 and Lemma 7.4.3, $\int \log|f'|\,d\mu = \int \log|g' \circ R^{-1}|\,d\mu$, the potential ψ is not cohomologous to a constant. The second part of Theorem 7.4.5 is now an immediate consequence of Theorem 7.2.1(b). ♣

Theorem 7.4.6. If $f : \partial\Omega \to \partial\Omega$ is a conformal boundary repeller, Ω is a Jordan domain and the functions $\log|g'|$ and $\log|f'| \circ R$ are cohomologous, (this is exactly the case of Theorem 7.4.5(b)), then Ω is an analytic curve. If additionally f extends analytically onto $\overline{\mathbb{C}}$, then f is analytically conjugate with a Möbius transformation and $\partial\Omega$ is a geometric circle.

Proof. If the functions $\log|g'|$ and $\log|f'| \circ R$ are cohomologous, then according to Theorem 7.4.5(a) the boundary $\partial\Omega$ is a rectifiable Jordan curve. So, in view of Riesz's theorem, the map $R : S^1 \to \partial\Omega$ transports the measure class of the Lebesgue measure on S^1 onto the measure class of H_1 on $\partial\Omega$. Let now $R_* : S^1 \to \partial\Omega$ be the restriction to the unit circle of the Riemann map induced by the second component of the complement of $\partial\Omega$. For technical reason, which will be clear at the end of this proof we assume here that the Riemann map R_* is defined on $D^1_* = \{z : |z| > 1\}$, the complement of the closure of the unit disk D^1. Since $\partial\Omega$ is a Jordan curve and since f has no critical points on $\partial\Omega$, there exists an open neighborhood A_* of $\partial\Omega$ such that $f(A_* \cap (\overline{\mathbb{C}} \setminus \partial\Omega))$ is well-defined, and moreover $f(A_* \cap \overline{\mathbb{C}} \setminus \partial\Omega) \subset \overline{\mathbb{C}} \setminus \partial\Omega)$. Therefore, we can define $g_* = R_*^{-1} \circ f \circ R_*$, the lift of f via the Riemann map R_* on the set D^1_* intersected with a sufficiently thin open annulus surronding S^1. Set

$$h = R_*^{-1} \circ R|_{S^1} : S^1 \to S^1.$$

Composing, if necessary, the Riemann maps R and R_* with appropriate rotations, we may assume that 1 is a fixed point of g and g_* and $h(1) = 1$. Our first objective is to demonstrate that h is real-analytic. Indeed, Let $\mu_1 = \phi_1 l$ and $\mu_2 = \phi_2 l$ be the (unique) probability measures equivalent with the Lebesgue measure on the circle, respectively invariant under the action of g and g_*. In view of Theorem 7.9.2 ϕ_1 and ϕ_2 are both Real-analytic. Since, also by Riesz's theorem, the map $R_* : S^1 \to \partial\Omega$ transports the measure class of the Lebesgue measure on S^1 onto the measure class of H_1 on $\partial\Omega$, the homeomorphism $h : S^1 \to S^1$ sends the measure class of the Lebesgue measure on S^1 onto itself. Since h establishes conjugacy between g and g_*, it therefore maps the invariant measure μ_1 onto some probability g_*-ivariant measure equivalent with the Lebesgue measure. Since such a measure is unique, it must be equal to μ_2. Symbolically, $\mu_1 \circ h^{-1} = \mu_2$. Define now two functions $M, N : S^1 \to [0,1]$ by setting

$$M(z) = \mu_1([1,z]) = \int_1^z \phi_1 \, dl$$

and

$$N(z) = \mu_2([1,z]) = \int_1^z \phi_2 \, dl.$$

Since $\mu_1 \circ h^{-1} = \mu_2$ and $h(1) = 1$, the functions M and N are related by the formula $N(h(z)) = M(z)$. Since M and N are strictly increasing, we may solve the last equation for h to get $h = M \circ N^{-1}$. We are now done because the real analyticity of ϕ_1 and ϕ_2 (see ???) implies that the functions M and N are real-analytic and this in turn results in real analyticity of $h = M \circ N^{-1}$. Hence h extends to an analytic map $\tilde{h}$ defined on an open neighborhood H of S^1 in $\mathbb{C}$. Since $h : S^1 \to S^1$ preserves orientation, decreasing H if necessary, we get $\tilde{h}(H \cap \overline{D^1_*}) \subset \overline{D^1_*}$. Thus we produced two continuous maps $R : \overline{D^1} \to \mathbb{C}$ and $R_* \circ \tilde{h} : H \cap \overline{D^1_*} \to \mathbb{C}$ which are analytic on D^1 and $H \cap D^1_*$ respectively and which coincide on their common boundary, the unit circle S^1. Thus R and $R_* \circ \tilde{h}$ glue together to an analytic map $S : D^1 \cup H \to \overline{\mathbb{C}}$. So, since $S(S^1) = R(S^1) = \partial\Omega$, the proof of the first part is finished.

If now f extends analytically to $\overline{\mathbb{C}}$, that is if f is a rational function, then by the maximum principle $f(\Omega) \subset \Omega$, $f(\overline{\mathbb{C}} \setminus \overline{\Omega}) \subset \overline{\mathbb{C}} \setminus \overline{\Omega}$, and both maps g and g_* are defined on D^1 and D^1_* respectively. Since these maps are surjective, since they extend continuously to $\overline{D^1}$ and $\overline{D^1_*}$ respectively, and since they preserve the unit circle S^1, it follows from the Schwarz reflection principle that they extend analytically to $\overline{\mathbb{C}}$. Since g and g^* preserve S^1 (we use this fact second time) they must be finite Blaschke products. Since $f(\Omega) \subset \Omega$ and $f(\overline{\mathbb{C}} \setminus \overline{\Omega}) \subset \overline{\mathbb{C}} \setminus \overline{\Omega}$, by the Montel theorem both Ω and $\overline{\mathbb{C}} \setminus \Omega$ are components of the Fatou set of f. And since $f|_{\partial\Omega}$ is expanding, both Ω and $\overline{\mathbb{C}} \setminus \overline{\Omega}$ are basins of immediate attraction to stable fixed points. Conjugating f if necessary with a Möbius transformation we may assume that this fixed point in D^1 is 0 and that one in D^1_* coincides with ∞. But every Blaschke product B preserving S^1 and having 0 as its fixed point, preserves the Lebesgue measure on S^1. In order to see it consider an arbitrary continuous function $\phi : S^1 \to \mathbb{R}$ and then its harmonic extension $\tilde{\phi} : D^1 \to \mathbb{R}$. Since $\phi \circ B$ is also harmonic, we have

$$\int_{S^1} \phi \circ B \, dl = \phi \circ B(0) = \phi(0) = \int_{S^1} \phi \, dl.$$

which means that B preserves the Lebesgue measure l. Thus, $\mu_1 = \mu_2 = l$ and consequently $M = N$ and h is the identity map on S^1. In conclusion $\tilde{h}$ is an identity map and R_* and $R = R_* \tilde{h}$ coincide on a neighborhood of S^1. Thus R and R_* glue together to an analytic map $\tilde{R} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Since $\tilde{R}$ is injective it must be a Möbius transformation and $\tilde{R}^{-1} \circ f \circ R^{-1}$ is a finite Blaschke product. The proof is finished. ♣

Section 7.5. Radial behaviour of the Riemann map,II.

Keeping notation from the previous sections we shall prove here the following .

Theorem 7.5.1. Depending on whether $c(\omega) = 0$ or $c(\omega) \neq 0$, either $\partial\Omega$ is real-analytic and the Riemann map $R : D^1 \to \Omega$ and its derivative R' extend holomorphically beyond ∂D^1 or for almost every $z \in \partial D^1$

$$\limsup_{r\to 1} |R'(rz)| \exp c\sqrt{\log(1/1-r)\log_3(1/1-r)} = \begin{cases} \infty & \text{if } c \leq c(\omega) \\ 0 & \text{if } c > c(\omega) \end{cases} \tag{7.5.1}$$

and

$$\limsup_{r\to 1} \big(|R'(rz)| \exp c\sqrt{\log(1/1-r)\log_3(1/1-r)}\big)^{-1} = \begin{cases} \infty & \text{if } c \leq c(\omega) \\ 0 & \text{if } c > c(\omega) \end{cases} \tag{7.5.2}$$

Moreover the radial limsup can be replaced by the nontangential one.

Proof. Let $n > 0$ be the least integer for which $g^n(rz) \in B(0, r_0)$ for some fixed $r_0 < 1$. We have $R'(rz) = ((f^n)'(R(rz)))^{-1} \cdot R'(g^n(rz)) \cdot (g^n)'(rz)$. Hence, for some constant $K > 0$ independent of r and z

$$K^{-1} \leq \frac{|R'(rz)|}{|((f^n)'(R(rz)))^{-1}| \cdot |R'(g^n(rz))|} \leq K.$$

By the bounded distortion theorem the rz in the denominator can be replaced z and n depends on r as described by (7.2.1) with r replaced by $1-r$.Now we proceed as in the proof of Lemma 7.2.1 replacing deviations of $S_n(\phi) - P(\phi)n + \kappa \log|(f^n)'(x)|$ by the deviations of $\log|(g^n)'(x)| - \log|(f^n)'(x)|$. The proof is finished. ♣

Section 7.6.. Pressure versus integral means of the Riemann map

In this section we establish a close relation between integral means of derivatives of the Riemann map and topological pressure of the function $-t\log|f'|$. Given $t \in I\!R$ define

$$\beta(t) = \limsup_{r\to 1} \frac{\log \int_{S^1} |R'(rz)|^t dl(z)}{-\log(1-r)}$$

We shall prove the following.

Theorem 7.6.1. If the lifted map $g : S^1 \to S^1$ is of the form $z \mapsto z^d$, $d \geq 2$, then

$$\beta(t) = t - 1 + \frac{\mathrm{P}(f, -t\log|f'|)}{\log d}.$$

Proof. Fix $0 < r < 1$ and divide the circle S^1 into $[2\pi/(1-r)]$ arcs of length $1-r$ and one arc of length $\le 1-r$. Denote these arcs by $I_1, I_2, \ldots, I_k$ and I_{k+1} respectively, where $k = [2\pi/(1-r)]$. Then

$$\int_{S^1} |R'(rz)|^t dl(z) = \sum_{j=1}^{k+1} \int_{I_j} |R'(rz)|^t dl(z) = \sum_{j=1}^{k+1} \int_{I_j} |R'(g^n(rz))|^t \frac{(|g^n)'(rz)|^t}{|(f^n)'(R(rz))|^t} dl(z).$$

Fix now $n = n(r)$ to be the first integer for which $|g^n(z)| < 1/2$. Note that n is independent of z and that there exists a constant $A \ge 1$ such that $A^{-1} \le |R'(w)| \le A$ for all $w \in B(0, 1/2)$. Hence

$$\int_{S^1} |R'(rz)|^t dl(z) \asymp \sum_{j=1}^{k+1} \int_{I_j} \frac{(|g^n)'(rz)|^t}{|(f^n)'(R(rz))|^t} dl(z). = \sum_{j=1}^{k+1} \int_{I_j} \frac{(d^{nt}}{|(f^n)'(R(rz))|^t} dl(z).$$

Now, by the Mean Value Theorem for every $j = 1, 2, \ldots, k, k+1$ there exists $z_j \in I_j$ such that

$$\int_{I_j} \frac{(d^{nt}}{|(f^n)'(R(rz))|^t} dl(z) = l(I_j) \frac{(d^{nt}}{|(f^n)'(R(rz_j))|^t}.$$

Hence, as $l(I_j) = 1-r$ for all $j = 1, \ldots, k$

$$\begin{aligned}
&\log \int_{S^1} |R'(rz)|^t dl(z) \\
&= \log(1-r) + \log\left(\sum_{j=1}^{k} \frac{(d^{nt}}{|(f^n)'(R(rz_j))|^t} + \frac{(d^{nt}}{|(f^n)'(R(rz_{k+1}))|^t} \cdot \frac{l(I_{k+1})^t}{(1-r)^t} \right) + O(1) \\
&= \log(1-r) + nt \log d + \\
&+ \log\left(\sum_{j=1}^{k} \exp \sum_{u=0}^{n-1} -t \log|f'(R(g^u(rz_j)))| + |(f^n)'(R(rz_j))|^t \frac{l(I_{k+1})^t}{(1-r)^t} \right) + O(1)
\end{aligned}$$

By our definition of n, $(1/2)^d \le r^{d^n} \le 1/2$; hence $d\log(1/2) \le d^n \log r \le \log(1/2)$. Since there exists a constant $B \ge 1$ such that $B^{-1}(1-r) \le -\log r \le B(1-r)$ for all r sufficiently close to 1, we get $B^{-1}\log 2 \le d^n(1-r) \le Bd\log 2$. Therefore $-\log B + \log\log 2 \le n\log d + \log(1-r) \le \log B + \log\log 2 + \log d$. Hence $n\log d - C \le -\log(1-r) \le n\log d + C$ for some universal constant C. Thus

$$\begin{aligned}
&\frac{\log \int_{S^1} |R'(rz)|^t dl(z)}{-\log(1-r)} = -1 + t + \\
&\frac{1}{n\log d} \log\left(\sum_{j=1}^{k} \exp \sum_{u=0}^{n-1} -t\log|f'(R(g^u(rz_j)))| + \exp \sum_{u=0}^{n-1} -t\log|f'(R(g^u(rz_{k+1})))| \frac{l(I_{k+1})^t}{(1-r)^t} \right) \\
&+ o(1).
\end{aligned}$$

Now, using the fact that $0 \le l(I_{k+1})/(1-r) \le 1$, it follows from the definition of pressure and the bounded distortion property that

$$\limsup_{r\to 1} \frac{\log \int_{S^1} |R'(rz)|^t dl(z)}{-\log(1-r)} = -1+t+\frac{1}{\log d}\mathrm{P}(g, -t\log|f'\circ R|) = -1+t+\frac{\mathrm{P}(f,-t\log|f'|)}{\log d}.$$

The proof is finished. ♣

Section 7.7. Geometric examples.

This last section of this chapter is devoted to explore applications of previous sections to geometric examples like the Koch's snowflake and Carleson's example. Following the idea of the proof of Theorem 7.2.1 and copeing with a biggere number of technicalities one can prove the following.

Theorem 7.7.1. Let Ω be a simply connected domain in $\mathbb{C}$ with $\partial\Omega$ a Jordan curve . Let ∂_j, $j = 1, 2 \ldots, k$ be a finite family of compact arcs in $\partial\Omega$ with pairwise disjoint interriors. Denote $\bigcup \partial_j$ by ∂ (we do not assume that this curve is connected. Assume that there exists a family of conformal maps f_j, $j = 1, \ldots, k$, (which may reverse the orientation on $\mathbb{C}$) on neighbourhoods U_j of ∂_j. For every j assume that $f_j(\Omega \cap U - J) \subset \Omega$, $|f'| > 1$ on U_j and

$$f_j(\partial\Omega \cap U_j) \subset \partial\Omega. \tag{7.7.1}$$

Assume also the Markov partition property: for every $j = 1, \ldots k$, $f_j(\partial_j) = \bigcup_{s\in I_j} \partial_s$ for some subset $I_j \subset \{1, 2, \ldots, k\}$. Consider the $k \times k$ matrix $A = A_{jk}$ where $A_{jk} = 1$ if $k \in I_j$ and $A_{jk} = 0$ if $k \notin I_j$. Then there exists a transition parameter $c(\omega, \partial)$ such that the claims of tTheorem 7.6.5 and 7.6.6 hold.

Example 1(*the snowflake*). To every side of an equilateral triangle, in the middle we glue from outside as small as three times. To every side of the resulting polygon we we glue again an equilateral triangle as small as three times and so on infinitely many times. The triangles do not overlap in this construction and the boundary of the resulting domain Ω is a Jordan curve.This Ω is called the Koch's snowflake. It was first describe by Helge Koch in 1904. Denote the curve in $\partial\Omega$ joining a point $x \in \partial\Omega$ to $y \in \partial\Omega$ in the clockwise direction just by xy. For every $\partial A_i A_{i+1(mod 12)} \subset \partial\Omega$, $i = 0, 1, \ldots, 11$, we consider its civering by the curves 12, 23, 45, 56 in Ω (see Fig.2). This covering together with the affine maps

12, 34 $\to$ 12 (preserving orientation on $\partial\Omega$)
23 $\to$ 61 (reversing orientation)
56 $\to$ 36 (preserving orientation)
45 $\to$ 63 (reversing orientation)

gives a Markov partition of ∂_i satisfying the assumptions of Theorem 6.7.1. Since $\partial\Omega$ (and every its subcurve) is definitely not real-analytic ($\mathrm{HD}(\partial\Omega) = \log 4/\log 3$), the assertion of Theorem 6.7.1 is valid with $c(\omega, \partial_i) > 0$. We may denote $c(\omega, \partial_i)$ by $c(\omega)$ since it is independent of ∂_i by symmetry.

Example 2(*Carleson's domain*). We recall Carleson's construction from [Ca2]. We fix a broken line γ with the first and last segment lying in the same straight line in $I\!R^2$, with no other segments intersecting the segment $\overline{1, d-1}$ (see Fig. 3). Then we take a regular polygon Ω^1 with vertices $T_0, T_1, \dots, T_n$ and glue to every side of it, from outside, the rescaled, not mirror reflected, curve γ so that the ends of the glued curve coincide with the ends of the side. The resulting curve bounds a second polygon Ω^2. Denote its vertices by $A_0, A_1, \dots$ (Fig. 4). Then we glue again the rescaled γ to all sides of Ω^2 and a third order polygon Ω^3 with vertices $B_0, B_1, \dots$. Then we bild Ω^4 with vertices $C_0, C_1, \dots \Omega^5$ with $D_0, D_1, \dots$ etc. Assume that there is no self-intersecting of the curves $\partial\Omega^n$ in this construction. Moreover assume that in the limit we obtain a Jordan curve $\mathcal{L} = \mathcal{L}(\Omega^1, \gamma) = \partial\Omega$. The natural Markov partition of each curve T_iT_{i+1}in $\mathcal{L}$ into curves A_jA_{j+1} with $f(A_jA_{j+1}) = T_iT_{i+1}$, considered by Carleson does not satisfy the property (6.7.1) so we cannot succeed with it. Instead we proceed as follows: Define in an affine fashion

$$f(B_{d(j-1)+1}B_{dj-1}) = A_1A_{d-1}$$

for every $j = 1, 2 \dots, d$. Divide now every arc $B_{dj-1}A_j$ for $j = 1, 2 \dots, d$ and A_jB_{dj+1}, $j = 1, 2 \dots, d$ into curves with ends in the vertices of the polygon Ω^4 : $C^j \in B_{dj_1}A_j$, $\tilde{C}^j \in A_jB_{dj+1}$ respectively, the closest to $A_j (\neq A_j)$. Let for $j = 1, 2, \dots, d-1$,

$$f(C^jA_j) = B_{dj-1}A_j, f(B_{dj-1}C^j) = A_{d-1}B_{d^2-1},$$
$$f(A_j\tilde{C}^j) = A_jB_{dj+1}, f(\tilde{C}^jB_{dj+1}) = B_1A_1.$$

This gives a transitive aperiodic Markov partition of $B_1B_{d^2-1}$. We can consider instead of the broken line γ in the construction of Ω, the line $\gamma^{(2)}$, consisting of d^2 segments, which arises by glueing to every side of γ a rescaled γ. Consecutive gluing of the rescaled $\gamma^{(2)}$ to the polygon $\Omega^{(1)}$ gives consecutively Ω^3, Ω_5 etc. The same construction as above gives a Markov partition of $D_1D_{d^2-1}$ in T_iT_{i+1}. By continuing this procedure we approximate T_iT_{i+1}, so from Theorem 6.7.1 and from symmetry we deduce that there exists a transition parameter $c(\omega)$ such that the assertion of Theorem 6.6.5(b) is satisfied. Observe that Carleson's assumption that the broken line $1, 2, \dots, d-1$ has no self-intersections has not been needed in these considerations. Also the assumption that $\Omega^{(1)}$ is a regular polygon can be omitted; one can prove that $c(\omega)$ doesnot depend on T_iT_{i+1} by considering a transitive, aperiodic Markov partition which involves all the sides of Ω^1 simultaneously.

Section 7.9. Real analyticity of the density functions. In this section we consider potentials of the form $-t\log|f'|$, fixed points of the corresponding conjugate transfer operators m_t and invariant Gibbs states μ_t. Our aim is to show that the Radon-Nikodym derivative $\frac{d\mu_t}{dm_t}$ has a real-analytic extension. We begin with the following.

Definition 7.9.1. A conformal expanding repeller $f : X \to X$ is said to be real-analytic if it is contained in a finite union of pairwise disjoint real-analytic curves which will be denoted by $\Gamma = \Gamma_f$. Frequently in such a context we will alternatively speak about real analyticity of the set X.

The main (and only) result of this section is the following.

Theorem 7.9.2. If $f : X \to X$ is an orientation preserving conformal expanding repeller, $X \subset C$, then the Radon-Nikodym derivative $\rho = d\mu_t/dm_t$ has a real-analytic real-valued extension on a neighbourhood of X in $\mathbb{C}$. If f is real-analytic, then ρ has a real-analytic extension on a neighbourhood of X in Γ.

Proof. Observe that since f is conformal and orientation preserving, f is holomorphic on a neighbourhood of X in $\mathbb{C}$. Take $r > 0$ so small that for every $x \in X$, every $n \geq 1$ and every $y \in f^{-n}(x)$ the holomorphic inverse branch $f_y^{-n} : B(x, 2r) \to \mathbb{C}$ sending x to y is well-defined. Suppose first that f is real-analytic. We need to show that there exists a holomorphic complex-valued extension of ρ on a neighbourhood of X in $\mathbb{C}$. Taking an appropriate atlas we may assume that X is contained in a real axis (if a closed curve is a component of Γ we can use Arg). For all $k \geq 1$ and all $y \in f^{-k}(x)$ let $\nu(k, y) = 1$ or -1 depending as f_y^{-k} preserves or reverses the orientation on Γ. So

$$|(f_y^{-k})'(z)| = \nu(k, y)((f_y^{-k})'(z))$$

for all $z \in J(f) \cap B(x, r)$. Consider the following sequence of complex analytic functions on $z \in B(x, r)$

$$g_n(z) = \sum_{y \in f^{-n}(x)} \Big(\nu(n, y)((f_y^{-n})'(z))\Big)^t \exp(-nP(t)).$$

There is no problem here with raising to the t-th power since $B(x, r)$, the domain of all $\nu(n, y)(f_y^{-n})'$ is simply connected. Since the latter functions are positive in $\mathbb{R}$, we can choose the branches of the t-th powers to be also positive in $\mathbb{R}$. By Koebe's Distortion Theorem for every $z \in B(x, r/2)$, every $n \geq 1$ and every $y \in f^{-n}(x)$ we have $|(f_y^{-n})'(z)| \leq K|(f_y^{-n})'(x)|$. Hence $|g_n(z)| \leq Kg_n(x)$. Since, by (3.4.2) with $u = 1$ and $c = p(t)$, the sequence $g_n(x)$ converges, we see that the functions $\{g_n|_{B(x,r/2)}\}_{n \geq 1}$ are uniformly bounded. So they form a normal family in the sense of Montel. Since $g_n(z)$ converges for all $z \in X \cap B(x, r/2)$, it follows that g_n converges to an analytic function g on $B(x, r/2)$ whose restriction to Γ is by our construction an extension of ρ.

Let us pass now to the proof of the first part of this proposition. That is, we relax the Julia real analyticity assumption and we want to construct a real-analytic real-valued extension of ρ to a neighbourhood of X in $\mathbb{C}$. Our strategy is to work in $\mathbb{C}^2$, to use an appropriate version of Montel's theorem and, in general, to proceed similarly as in the first part of the proof. So, fix $v \in X$. Identify now $\mathbb{C}$, where our f acts, to $\mathbb{R}^2$ with coordinates x, y, the real and complex part of z. Embed this into $\mathbb{C}^2$ with x, y complex. Denote the above

$\mathbb{C} = \mathbb{R}^2$ by $\mathbb{C}_0$. We may assume that $v = 0$ in $\mathbb{C}_0$. Given $k \geq 0$ and $v_k \in f^{-k}(v)$ define the function $\rho_{v_k} : B_{\mathbb{C}_0}(0, 2r) \to \mathbb{C}$ (the ball in $\mathbb{C}_0$) by setting

$$\rho_{v_k}(z) = \frac{(f_{v_k}^{-k})'(z)}{(f_{v_k}^{-k})'(0)},$$

Since $B_{\mathbb{C}_0}(0, 2r) \subset \mathbb{C}_0$ is simply connected and ρ_{v_n} nowhere vanishes, all the branches of logarithm $\log \rho_{v_n}$ are well defined on $B_{\mathbb{C}_0}(0, 2r)$. Choose this branch that maps 0 to 0 and denote it also by $\log \rho_{v_n}$. By Koebe's Distortion Theorem $|\rho_{v_k}|$ and $|\mathrm{Arg}\rho_{v_k}|$ are bouned on $B(0, r)$ by universal constants K_1, K_2 respectively. Hence $|\log \rho_{v_k}| \leq K = (\log K_1) + K_2$. We write

$$\log \rho_{v_k} = \sum_{m=0}^{\infty} a_m z^m$$

and note that by Cauchy's inequalities

$$|a_m| \leq K/r^m. \tag{7.9.1}$$

We can write for $z = x + iy$ in $\mathbb{C}_0$

$$\mathrm{Re} \log \rho_{v_k} = \mathrm{Re} \sum_{m=0}^{\infty} a_m (x + iy)^m = \sum_{p,q=0}^{\infty} \mathrm{Re}\Big(a_{p+q} \binom{p+q}{q} i^q\Big) x^p y^q := \sum c_{p,q} x^p y^q.$$

In view of (2.1), we can estimate $|c_{p,q}| \leq |a_{p+q}| 2^{p+q} \leq K r^{-(p+q)} 2^{p+q}$. Hence $\mathrm{Re} \log \rho_{v_k}$ extends, by the same power series expansion $\sum c_{p,q} x^p y^q$, to the polydisc $D_{\mathbb{C}^2}(0, r/2)$ and its absolute value is bounded there from above by K. Now for every $k \geq 0$ consider a real-analytic function b_k on $B_{\mathbb{C}_0}(0, 2r)$ by setting

$$b_k(z) = \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(z)|^t \exp(-kP(t)).$$

By (3.4.2) the sequence $b_k(0)$ is bounded from above by a constant L. Each function b_k extends to the function

$$B_k(z) = \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{t \mathrm{Re} \log \rho_{v_k}(z)} \exp(-kP(t)).$$

whose domain, similarly as the domains of the functions $\mathrm{Re} \log \rho_{v_k}$, contains the polydisc $D_{\mathbb{C}^2}(0, r/2)$. Finally we get for all $k \geq 0$ and all $z \in D_{\mathbb{C}^2}(0, r/4)$

$$\begin{aligned} |B_k(z)| &= \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{\mathrm{Re}(t \mathrm{Re} \log \rho_{v_k}(z))} \exp(-kP(t)) \\ &\leq \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{t |\mathrm{Re} \log \rho_{v_k}(z)|} \exp(-kP(t)) \\ &\leq \mathrm{e}^{Kt} \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \exp(-kP(t)) \leq \mathrm{e}^{Kt} L. \end{aligned}$$

Now by Cauchy's integral formula (in $D_{\mathbb{C}^2}(0, r/4)$) for the second derivatives we prove that the family B_n is equicontinuous on, say, $D_{\mathbb{C}^2}(0, r/5)$. Hence we can choose a uniformly convergent subsequence and the limit function G is complex analytic and extends ρ on $X \cap B(0, r/5)$, by (3.4.2). Thus we have proved that ρ extends to a complex analytic function in a neighbourhood of every $v \in X$ in $\mathbb{C}^2$, i.e. real analytic in $\mathbb{C}_0$. These extensions coincide on the intersections of the neighbourhoods, otherwise X is real analytic and we are in the case considered at the beginning of the proof. ♣