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CHAPTER 9
CONFORMAL MAPS WITH INVARIANT PROBABILITY MEASURES OF POSITIVE LYAPUNOV EXPONENT.

§9.1 RUELLE'S INEQUALITY.

Let X be a compact subset of the closed complex plane $\overline{\mathbb{C}}$ and let $\mathcal{A}(X)$ denote the set of all continuous maps $f : X \to X$ that can be analytically extended to an open neighbourhood $U(f)$ of X. In this section we only work with the standard spherical metric on $\overline{\mathbb{C}}$, normalized so that the area of $\overline{\mathbb{C}}$ is 1. In particular all the derivatives are computed with respect to this metric.

Let us recall and extend Definition 6.1.3. Let μ be an f-invariant Borel probability measure on X. Since $|f'|$ is bounded, the integral $\int \log |f'| \, d\mu$ is well-defined and moreover $\int \log |f'| \, d\mu < +\infty$. The number

$$\chi_\mu = \chi_\mu(f) = \int \log |f'| \, d\mu$$

is called the *Lyapunov characteristic exponent* of μ and f. Note that $\int \log |f'| d\mu = -\infty$ is not excluded. In fact it is possible, for example if $X = \{0\}$ and $f(z) = z^2$.

By Birkhoff Ergodic Theorem (Th 1.2.2) the *Lyapunov characteristic exponent* $\chi_\mu(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|$ exists for a.e. x, compare Sec.6, and $\int \chi_\mu(x) \, d\mu(x) = \chi_\mu$. (In fact one allows $\log |f'|$ with integral $-\infty$ here, so one need extend slightly Th.1.2.2. This is not difficult.)

The section is devoted to prove the following.

Theorem 9.1.1. (Ruelle's inequality) If $f \in \mathcal{A}(X)$, then $\mathrm{h}_\mu(f) \le 2 \int \max\{0, \chi_\mu(x)\} \, d\mu$. For ergodic μ this yields $\mathrm{h}_\mu(f) \le 2 \max\{0, \chi_\mu\}$.

Proof. Consider a sequence of positive numbers $a_k \searrow 0$, and $\mathcal{P}_k$, $k = 1, 2, \ldots$ an increasing sequence of partitions of the sphere $\overline{\mathbb{C}}$ consisting of elements of diameters $\le a_k$ and of (spherical) areas $\ge \frac{1}{4} a_k^2$. Check that such partitions exist.

For every $g \in \mathcal{A}(X)$, $x \in X$ and $k \ge 1$ let

$$N(g, x, k) = \#\{P \in \mathcal{P}_k : g(P_k(x) \cap U(g)) \cap P \neq \emptyset\}$$

Our first aim is to show that for every $k > k(g)$ large enough

$$N(g, x, k) \le 4\pi(|g'(x)| + 2)^2 \tag{9.1.1}$$

Indeed, fix $x \in X$ and consider k so large that $\mathcal{P}_k(x) \subset U(g)$ and a Lipschitz constant of $g|_{\mathcal{P}_k(x)}$ does not exceed $|g'(x)|+1$. Thus the set $g(\mathcal{P}_k(x))$ is contained in the ball $B(g(x),(|g'(x)|+1)a_k)$. Therefore if $g(\mathcal{P}_k(x)) \cap P \neq \emptyset$, then

$$P \subset B(g(x),(|g'(x)|+1)a_k + a_k) = B(g(x),(|g'(x)|+2)a_k)$$

Hence $N(g,x,k) \leq \pi(|g'(x)|+2)^2 a_k^2 / \frac{1}{4} a_k^2 = 4\pi(|g'(x)|+2)^2$ and (9.1.1) is proved.

Let $N(g,x) = \sup_{k > k(g)} N(g,x,k)$. In view of (9.1.1) we get

$$N(g,x) \leq 4\pi(|g'(x)|+2)^2 \tag{9.1.2}$$

Now note that for every finite partition $\mathcal{A}$ one has

$$\mathrm{h}(g,\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n+1}\mathrm{H}(\mathcal{A}^n) = \lim_{n\to\infty} \frac{1}{n+1}\Big(\mathrm{H}(g^{-n}(\mathcal{A})|\mathcal{A}^{n-1}) + \ldots + \mathrm{H}(g^{-1}(\mathcal{A})|\mathcal{A}) + \mathrm{H}(\mathcal{A})\Big)$$

$$\leq \lim_{n\to\infty} \frac{1}{n}\Big(\mathrm{H}(g^{-n}(\mathcal{A})|g^{-(n-1)}(\mathcal{A})) + \ldots + \mathrm{H}(g^{-1}(\mathcal{A})|\mathcal{A})\Big) = \mathrm{H}(g^{-1}(\mathcal{A})|\mathcal{A}). \tag{9.1.3}$$

(Compare this computation with the one done in Theorem 1.4.5 or in Proof of Theorem 1.5.4, which would result with $\mathrm{h}(g,\mathcal{A}) \leq \mathrm{H}(\mathcal{A}|g^{-1}(\mathcal{A}))$.) Going back to our situation, since

$$\mathrm{H}_{\mu_{\mathcal{P}_k(x)}}(g^{-1}(\mathcal{P}_k)|\mathcal{P}_k(x)) \leq \log \#\{P \in \mathcal{P}_k : g^{-1}(P) \cap \mathcal{P}_k(x) \neq \emptyset\} = \log N(g,x,k)$$

and by Theorem 1.8.7.a, we obtain

$$\mathrm{h}_\mu(g) \leq \limsup_{k\to\infty} \mathrm{H}_\mu(g^{-1}(\mathcal{P}_k)|\mathcal{P}_k) = \limsup_{k\to\infty} \int \mathrm{H}_{\mu_{\mathcal{P}_k(x)}}(g^{-1}(\mathcal{P}_k)|\mathcal{P}_k(x))\, d\mu(x)$$

$$\leq \limsup_{k\to\infty} \int \log N(g,x,k)\, d\mu(x) \leq \int \log N(g,x)\, d\mu(x).$$

Applying this inequality to $g = f^n$ ($n \geq 1$ an integer) and employing (9.1.2) we get

$$\begin{aligned}\mathrm{h}_\mu(f) &= \frac{1}{n}\mathrm{h}_\mu(f^n) \leq \frac{1}{n}\int \log N(f^n,x)\, d\mu(x) = \int \frac{1}{n}\log N(f^n,x)\, d\mu(x) \\ &\leq \int \frac{1}{n}\log 4\pi(|(f^n)'(x)|+2)^2\, d\mu(x)\end{aligned}$$

Since $0 \leq \frac{1}{n}\log(|(f^n)'(x)|+2)^2 \leq 2(\log\big(\sup_X |f'|\big)+1)$ and $\lim_{n\to\infty}\frac{1}{n}\log(|(f^n)'(x)|+2) = \max\{0,\chi_\mu(x)\}$ for μ-a.e $x \in X$, it follows from the Dominated Convergence Theorem (Ch1.Sec.1) that

$$h_\mu(f) \leq \lim_{n\to\infty} \int \frac{1}{n}\log(|(f^n)'(x)|+2)^2\, d\mu(x) = \int \max\{0, 2\chi_\mu(x)\}\, d\mu.$$

The proof is completed. ♣

Exercise. Prove the following general version of Theorem 8.1.1: Let X be a compact f-invariant subset of a smooth Riemannian manifold for a C^1 mapping $f : U \to M$, defined on a neighbourhood U of X. Let μ be an f-invariant Borel probability measure X. Then

$$h_\mu(f) \le \int_X \max\{0, \chi_\mu^+(x)\}\, d\mu(x),$$

where $\chi_\mu^+(x) = \lim_{n\to\infty} \frac{1}{n} \log \|(Df^n)^\wedge\|$. Here Df^n is the differential and $(Df^n)^\wedge$ is the exterior power, the linear operator between the exterior algebras generated by the tangent spaces at x and $f^n(x)$. The norm is induced by the Riemann metric. Saying directly $\|(Df^n)^\wedge\|$ is supremum of the volumes of Df^n-images of unit cubes in k-dimensional subspaces of T_xM with $k = 0, 1, ..., \dim M$.

Note. Theorem 9.1.1 and Exercise rely on [Ruelle] D. Ruelle: An inequality of the entropy of differentiable maps. Bol. Soc. Bras. Mat. 9 (1978), 83-87.

§9.2. PESIN'S THEORY

In this section we work in the same setting and we follow the same notation as in Section 9.1.

Lemma 9.2.1. If μ is a Borel finite measure on $I\!R^n$, $n \ge 1$, a is an arbitrary point of $I\!R^n$ and the function $z \mapsto \log|z-a|$ is μ-integrable, then for every $C > 0$ and every $0 < t < 1$,

$$\sum_{n\ge1} \mu(B(a, Ct^n)) < \infty.$$

Proof. Since μ is finite and since given $t < s < 1$ there exists $q \ge 1$ such that $Ct^n \le s^n$ for all $n \ge q$, without loosing generality we may assume that $C = 1$. Recall that given $b \in I\!R^n$, and two numbers $0 \le r < R$, $R(b, r, R) = \{z \in \mathbb{C} : r \le |z-b| < R\}$. Since $-\log(t^n) \le -\log|z-a|$ for every $z \in B(a, t^n)$ we get the following.

$$\begin{aligned}\sum_{n\ge1} \mu(B(a,t^n)) &= \sum_{n\ge1} n\mu(R(a,t^{n+1},t^n)) = \frac{-1}{\log t}\sum_{n\ge1} -\log(t^n)\mu(R(a,t^{n+1},t^n)) \\ &\le \frac{-1}{\log t}\int_{B(a,t)} -\log|z-a|\, d\mu(z) < +\infty\end{aligned}$$

The proof is finished. ♣

Lemma 9.2.2. If μ is a Borel finite measure on $\overline{\mathbb{C}}$, $n \ge 1$, and $\log|f'|$ is μ integrable, then the function $z \mapsto \log|z-c| \in L^1(\mu)$ for every critical point c of f. If additionally μ is f-invariant, then also the function $z \mapsto \log|z-f(c)| \in L^1(\mu)$.

Proof. That $\log|z-c| \in L^1(\mu)$ follows from the fact that near c we have $C^{-1}|z-q|^{q-1} \le |f'(z)| \le C|z-c|^{q-1}$, where $q \ge 2$ is the order of the critical point c and $C \ge 1$ is a universal constant, and since out of any neighbourhood of the set of critical points of f, $|f'(z)|$ is uniformly bounded away from zero and infinity. In order to prove the second part of the lemma, consider a ray R emanating from $f(c)$ such that $\mu(R) = 0$ and a disk $B(f(c), r)$ such that $f_c^{-1} : B(f(c), r) \setminus R \to \overline{\mathbb{C}}$, an inverse branch of f sending $f(c)$ to c, is well-defined. Let $D = B(f(c), r) \setminus R$. We may additionally require $r > 0$ to be so small that $|z - f(c)| \asymp |f_c^{-1}(z) - c|^q$. It suffices to show that the integral $\int_D \log|z - f(c)|\, d\mu(z)$ is finite. And indeed, by f-invariance of μ we have

$$\int_D \log|z-f(c)|\, d\mu(z) = \int_X 1_D(z) \log|z - f(c)|\, d\mu(z) \asymp \int_X 1_D(z) \log|f_c^{-1}(z) - c|^q\, d\mu(z)$$
$$= \int_X (1_D \circ f)(z) \log|z-c|^q\, d\mu(z) = \int_X 1_{f^{-1}(D)} \log|z-c|^q\, d\mu(z)$$

Notice here that the function $1_D(z)\log|f_c^{-1}(z) - c|^q$ is well-defined on X indeed and that unlike most of our comparability signs, the sign in the formula above means an additive comparability. The finiteness of the last integral follows from the first part of this lemma. ♣

Theorem 9.2.3. Let $(Z, \mathcal{F}, \nu)$ be a measure space with an ergodic measure preserving automorphism $T : Z \to Z$. Let $f : X \to X$ be a continuous map from a compact set $X \subset \overline{\mathbb{C}}$ onto itself having a holomorphic extention onto a neighbourhood of X ($f \in \mathcal{A}(X)$). Suppose that μ is an f-invariant ergodic measure on X with positive Lyapunov exponent. Suppose also that $h : Z \to X$ is a measurable mapping such that $\nu \circ h^{-1} = \mu$ and $h \circ T = f \circ h$ ν-a.e.. Then for ν-a.e. $z \in Z$ there exists $r(z) > 0$ such that for every $n \ge 1$ there exists $f_{x_n}^{-n} : B(x, r(z)) \to \overline{\mathbb{C}}$, an inverse branch of f^n sending $x = h(z)$ to $x_n = h(T^{-n}(z))$. In addition, for an arbitrary χ, $-\chi_\mu(f) < \chi < 0$, (not depending on z) and a constant $K(z)$

$$|(f_{x_n}^{-n})'(y)| < K(z) e^{\chi n} \text{ and } \frac{|(f_{x_n}^{-n})'(w)|}{|(f_{x_n}^{-n})'(y)|} \le K$$

for all $y, w \in B(x, r(z))$. K is here the Koebe constant corresponding to the scale $1/2$.

Proof. Suppose first that $\mu\big(\bigcup_{n\ge1} f^n(\mathrm{Crit}(f))\big) > 0$. Since μ is ergodic this implies that μ must be concentrated on a periodic orbit of an element $w \in \bigcup_{n\ge1} f^n(\mathrm{Crit}(f))$. This means that $w = f^q(c) = f^{q+k}(c)$ for some $q, k \ge 1$ and $c \in \mathrm{Crit}(f)$, and

$$\mu(\{f^q(c), f^{q+1}(c), \dots, f^{q+k-1}(c)\}) = 1.$$

Since $\int \log|f'|\, d\mu > 0$, $|(f^k)'(f^q(c))| > 1$. Thus the theorem is obviously true for the set $h^{-1}(\{f^q(c), f^{q+1}(c), \dots, f^{q+k-1}(c)\})$ of ν measure 1.

So, suppose that $\mu\big(\bigcup_{n\ge1} f^n(\mathrm{Crit}(f))\big) = 0$. Set $R = \min\{1, \mathrm{dist}(X, \overline{\mathbb{C}} \setminus U(f))\}$ and fix $\lambda \in (e^{\frac{1}{4}\chi}, 1)$. Consider $z \in Z$ such that $x = h(z) \notin \bigcup_{n\ge1} f^n(\mathrm{Crit}(f))$,

$$\lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(h(T^{-n}(z))| = \chi_\mu(f),$$

and $x_n = h(T^{-n}(z)) \in B(f(\mathrm{Crit}(f)), R\lambda^n)$ only for finitely many n's. We shall first demonstrate that the set of points satisfying these properties is of full measure ν. Indeed, the first requirement is satisfied by our hyphothesis, the second is due to Birkhoff's ergodic theorem. In order to prove that the set of points satisfying the third condition has ν measure 1 notice that

$$\begin{aligned}\sum_{n\ge 1} \nu\big(T^n(h^{-1}(B(f(\mathrm{Crit}(f)), R\lambda^n)))\big) &= \sum_{n\ge 1} \nu\big(h^{-1}(B(f(\mathrm{Crit}(f)), R\lambda^n))\big) \\ &= \sum_{n\ge 1} \mu(B(f(\mathrm{Crit}(f)), R\lambda^n)) < \infty,\end{aligned}$$

where the last inequality we wrote due to Lemma 9.2.2 and Lemma 9.2.1. The application of the Borel-Canteli lemma finishes now the demonstration. Fix now an integer $n_1 = n_1(z)$ so large that $x_n = h(T^{-n}(z)) \notin B(f(\mathrm{Crit}(f)), R\lambda^n)$ for all $n \ge n_1$. Notice that because of our choices there exists $n_2 \ge n_1$ such that $|(f^n)'(x_n)|^{-1/4} < \lambda^n$ for all $n \ge n_2$. Finally set $S = \sum_{n\ge 1} |(f^n)'(x_n)|^{-1/4}$, $b_n = \frac{1}{2}S^{-1}|(f^{n+1})'(x_{n+1})|^{\frac{-1}{4}}$, and

$$\Pi = \Pi_{n=1}^{\infty}(1-b_n)^{-1}$$

which converges since the series $\sum_{n\ge 1} b_n$ converges. Choose now $r = r(z)$ so small that $16r(z)\Pi K S^3 \le R$, all the inverse branches $f_{x_n}^{-n} : B(x_0, \Pi r(z)) \to \overline{\mathbb{C}}$ are well-defined for all $n = 1, 2, \ldots, n_2$ and $\mathrm{diam}\big(f_{x_{n_2}}^{-n_2}(B\big(x_0, r\Pi_{k\ge n_2}(1-b_k)^{-1}\big)\big) \le \lambda^{n_2} R$. We shall show by induction that for every $n \ge n_2$ there exists an analytic inverse branch $f_{x_n}^{-n} : B\big(x_0, r\Pi_{k\ge n}(1-b_k)^{-1}\big) \to \mathbb{C}$, sending x_0 to x_n and such that

$$\mathrm{diam}\big(f_{x_n}^{-n}(B\big(x_0, r\Pi_{k\ge n}(1-b_k)^{-1}\big)\big) \le \lambda^n R.$$

Indeed, for $n = n_2$ this immediately follows from our requirements imposed on $r(z)$. So, suppose that the claim is true for some $n \ge n_2$. Since $x_n = f_{x_n}^{-n}(x_0) \notin B(\mathrm{Crit}(f), R\lambda^n)$ and since $\lambda^n R \le R$, there exists an inverse branch $f_{x_{n+1}}^{-1} : B(x_n, \lambda^n R) \to \mathbb{C}$ sending x_n to x_{n+1}. Since $\mathrm{diam}\big(f_{x_n}^{-n}(B\big((x_0, r\Pi_{k\ge n}(1-b_k)^{-1}\big)\big) \le \lambda^n R$, the composition $f_{x_{n+1}}^{-1} \circ f_{x_n}^{-n} B\big(x_0, r\Pi_{k\ge n}(1-b_k)^{-1}\big) \to \mathbb{C}$ is well-defined and forms the inverse branch of f^{n+1} that sends x_0 to x_{n+1}. By the Koebe distortion theorem we now estimate

$$\begin{aligned}&\mathrm{diam}\big(f_{x_{n+1}}^{-(n+1)}(B\big(x_0, r\Pi_{k\ge n+1}(1-b_k)^{-1}\big)\big)\big) \\ &\qquad \le 2r\Pi_{k\ge n+1}(1-b_k)^{-1}|(f^{n+1})'(x_{n+1})|^{-1}Kb_n^{-3} \\ &\qquad \le 16r\Pi K S^3 |(f^{n+1})'(x_{n+1})|^{-1}|(f^{n+1})'(x_{n+1})|^{\frac{3}{4}} \\ &\qquad = 16r\Pi K S^3 |(f^{n+1})'(x_{n+1})|^{-\frac{1}{4}} \\ &\qquad \le R\lambda^{n+1},\end{aligned}$$

where the last inequality sign we wrote due to our choice of r and the number n_2. Putting $r(z) = r/2$ the second part of this theorem follows now as a combined application of the equality $\lim_{n\to\infty} \frac{1}{n}\log|(f^n)'(x_n)| = \chi_\mu(f)$ and the Koebe distortion theorem. ♣

As an immediate consequence of Theorem 9.2.3 we get the following.

Corollary 9.2.4. Assume the same notation and asumptions as in Theorem 9.2.3. Fix $\varepsilon > 0$. Then there exist a set $Z(\varepsilon) \subset Z$, the numbers $r(\varepsilon) \in (0,1)$ and $K(\varepsilon) \geq 1$ such that $\mu(Z(\varepsilon)) > 1-\varepsilon$, $r(z) \geq r(\varepsilon)$ for all $z \in Z(\varepsilon)$ and with $x_n = h(T^{-n}(z))$

$$K(\varepsilon)^{-1}\exp(-(\chi_\mu+\varepsilon)n) \leq |(f_{x_n}^{-n})(y)| \leq K(\varepsilon)\exp(-(\chi_\mu-\varepsilon)n) \text{ and } \frac{|(f_{x_n}^{-n})'(w)|}{|(f_{x_n}^{-n})'(y)|} \leq K$$

for all $n \geq 1$, all $z \in Z(\varepsilon)$ and all $y, w \in B(x_0, r(\varepsilon))$. K is here the Koebe constant corresponding to the scale 1/2.

Remark 9.2.5. In our future applications the system (Z, f, ν) will be usually given by the natural extension of the holomorphic system (f, μ).

§9.3 MAÑÉ'S PARTITION

In this section, basically following Mañé's book ???, we construct so called Mañé's partition which will play an important role in the proof of a part of the Volume Lemma given in the next section. We begin with the following elementary fact.

Lemma 9.3.1. If $x_n \in (0,1)$ for every $n \geq 1$ and $\sum_{n=1}^\infty n x_n < \infty$, then $\sum_{n=1}^\infty -x_n \log x_n < \infty$.

Proof. Let $S = \{n : -\log x_n \geq n\}$. Then

$$\sum_{n=1}^\infty -x_n\log x_n = \sum_{n\notin S} -x_n\log x_n + \sum_{n\in S} -x_n\log x_n \leq \sum_{n=1}^\infty n x_n + \sum_{n\in S} -x_n\log x_n$$

Since $n \in S$ means that $x_n \leq e^{-n}$ and since $\log t \leq 2\sqrt{t}$ for all $t \geq 1$, we have

$$\sum_{n\in S} x_n \log\frac{1}{x_n} \leq 2\sum_{n=1}^\infty x_n\sqrt{\frac{1}{x_n}} \leq 2\sum_{n=1}^\infty e^{-\frac{1}{2}n} < \infty$$

The proof is finished. ♣

The next lemma is the main and simultaneously the last result of this section.

Lemma 9.3.2. If μ is a Borel probability measure concentrated on a bounded subset M of a Euclidean space and $\rho : M \to (0,1]$ is a measurable function such that $\log\rho$ is integrable with respect to μ, then there exists a countable measurable partition, called Mañé's partition, $\mathcal{P}$ of M such that $\mathrm{H}_\mu(\mathcal{P}) < \infty$ and

$$\mathrm{diam}(\mathcal{P}(x)) \le \rho(x)$$

for μ-almost every $x \in M$.

Proof. Let q be the dimension of the Euclidean space containing M. Since M is bounded, there exists a constant $C > 0$ such that for every $0 < r < 1$ there exists a partition $\mathcal{P}_r$ of M of diameter $\le r$ and which consists of at most Cr^{-q} elements. For every $n \ge 0$ put $U_n = \{x \in M : e^{-(n+1)} < \rho(x) \le e^{-n}\}$. Since $\log \rho$ is a non-positive integrable function, we have

$$\sum_{n=1}^{\infty} -n\mu(U_n) \ge \sum_{n=1}^{\infty} \int_{U_n} \log \rho \, d\mu = \int_M \log \rho \, d\mu > -\infty$$

so that

$$\sum_{n=1}^{\infty} n\mu(U_n) < +\infty. \tag{9.3.1}$$

Define now $\mathcal{P}$ as the partition whose atoms are of the form $Q \cap U_n$, where $n \ge 0$ and $Q \in \mathcal{P}_{r_n}$, $r_n = e^{-(n+1)}$. Then

$$\mathrm{H}_\mu(\mathcal{P}) = \sum_{n=0}^{\infty} \Big(- \sum_{U_n \supset P \in \mathcal{P}} \mu(P) \log \mu(P) \Big).$$

But for every $n \ge 0$

$$\begin{aligned}
- \sum_{U_n \supset P \in \mathcal{P}} \mu(P) \log \mu(P) &= \mu(U_n) \sum_P -\frac{\mu(P)}{\mu(U_n)} \log\Big(\frac{\mu(P)}{\mu(U_n)}\Big) - \mu(U_n) \sum_P \frac{\mu(P)}{\mu(U_n)} \log(\mu(U_n)) \\
&\le \mu(U_n)(\log C - q \log r_n) - \mu(U_n) \log \mu(U_n) \\
&\le \mu(U_n) \log C + q(n+1)\mu(U_n) - \mu(U_n) \log \mu(U_n).
\end{aligned}$$

Thus, summing over all $n \ge 0$, we obtain

$$\mathrm{H}_\mu(\mathcal{P}) \le \log C + q + q \sum_{n=0}^{\infty} n\mu(U_n) + \sum_{n=0}^{\infty} -\mu(U_n) \log \mu(U_n).$$

Therefore looking at (9.3.1) and Lemma 9.3.1 we conclude that $\mathrm{H}_\mu(\mathcal{P})$ is finite. Also, if $x \in U_n$, then the atom $\mathcal{P}(x)$ is contained in some atom of P_{r_n} and therefore

$$\mathrm{diam}(\mathcal{P}(x)) \le r_n = e^{-(n+1)} < \rho(x).$$

Now the remark that the union of all the sets U_n is of measure 1 completes the proof. ♣

§9.4 VOLUME LEMMA AND THE FORMULA $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f)$

In this section we keep the notation of Sections 9.1 and 9.2 and our main purpose is to prove the following two results which generalize the respective results in Chapter 7.

Theorem 9.4.1. If $f \in \mathcal{A}(X)$ and μ is an ergodic f-invariant measure with positive Lyapunov exponent, then $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f)$.

Theorem 9.4.2. (Volume Lemma) With the assumptions of Theorem 9.4.1

$$\lim_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log r} = \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)}$$

for μ-a.e. $x \in X$.

In view of Corollary 6.6.4, Theorem 9.4.1 follows from Theorem 9.4.2 and we only need to prove the latter one. Let us prove first

$$\liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log r} \ge \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)} \tag{9.4.1}$$

for μ-a.e. $x \in X$. By Corollary 7.1.9 there exists a finite partition $\mathcal{P}$ such that for an arbitrary $\varepsilon > 0$ and every x in a set X_o of full measure μ there exists $n(x) \ge 0$ such that for all $n \ge n(x)$.

$$B(f^n(x), e^{-\varepsilon n}) \subset \mathcal{P}(f^n(x)). \tag{9.4.2}$$

Let us work from now on in the natural extension $(\tilde X, \tilde f, \tilde\mu)$. Let $\tilde X(\varepsilon)$ and $r(\varepsilon)$ be given by Corollary 9.2.4, i.e. $\tilde X(\varepsilon) = Z(\varepsilon)$. In view of Birkhoff's Ergodic Theorem there exists a measurable set $\tilde F(\varepsilon) \subset \tilde X(\varepsilon)$ such that $\tilde\mu(\tilde F(\varepsilon)) = \tilde\mu(\tilde X(\varepsilon))$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \chi_{\tilde X(\varepsilon)} \circ \tilde f^n(\tilde x) = \tilde\mu(\tilde X(\varepsilon))$$

for every $\tilde x \in \tilde F(\varepsilon)$. Let $F(\varepsilon) = \pi(\tilde F(\varepsilon))$. Then $\mu(F(\varepsilon)) = \tilde\mu(\pi^{-1}(F(\varepsilon)) \ge \tilde\mu(\tilde F(\varepsilon)) = \tilde\mu(\tilde X(\varepsilon))$ converges to 1 if $\varepsilon \searrow 0$. Consider now $x \in F(\varepsilon) \cap X_o$ and take $\tilde x \in \tilde F(\varepsilon)$ such that $x = \pi(\tilde x)$. Then by the above there exists an increasing sequence $\{n_k = n_k(x) : k \ge 1\}$ such that $\tilde f^{n_k}(\tilde x) \in \tilde X(\varepsilon)$ and

$$\frac{n_{k+1} - n_k}{n_k} \le \varepsilon \tag{9.4.3}$$

for every $k \ge 1$. Moreover, we can assume that $n_1 \ge n(x)$. Consider now an integer $n \ge n_1$ and the ball $B\big(x, Cr(\varepsilon)\exp(-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)\big)$, where $0 < C < (Kr(\varepsilon))^{-1}$ is a constant (possibly depending on x) so small that

$$f^q\big(B(x, Cr(\varepsilon)\exp -(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)\big) \subset P(f^q(x)) \tag{9.4.4}$$

for every $q \le n_1$ and $K(\varepsilon) \ge 1$ is the constant appearing in Corollary 9.2.4. Take now any q, $n_1 \le q \le n$, and associate k such that $n_k \le q \le n_{k+1}$. Since $\tilde f^{n_k}(\tilde x) \in \tilde X(\varepsilon)$ and since $\pi(\tilde f^{n_k}(\tilde x)) = f^{n_k}(x)$, Corollary 9.2.4 produces a holomorphic inverse branch $f_x^{-n_k} : B(f^{n_k}(x), r(\varepsilon)) \to \overline{\mathbb{C}}$ of f^{n_k} such that $f_x^{-n_k}(f^{n_k}(x)) = x$ and

$$f_x^{-n_k}\big(B(f^{n_k}(x), r(\varepsilon))\big) \supset B\big(x, K(\varepsilon)r(\varepsilon)^{-1}\exp(-(\chi_\mu+\varepsilon)n_k)\big).$$

Since $B(x, Cr(\varepsilon)\exp-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n) \subset B\big(x, K(\varepsilon)^{-1}r(\varepsilon)\exp-(\chi_\mu+\varepsilon)n_k)\big)$, it follows from Corollary 9.2.4 that

$$\begin{aligned} f^{n_k}\big(B(x, Cr(\varepsilon)\exp-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)\big) \subset \\ \subset B\big(f^{n_k}(x), CKr(\varepsilon)e^{-\chi_\mu(n-n_k)}\exp(\varepsilon(n_k - (2+\log\|f'\|)n))\big). \end{aligned}$$

Since $n \ge n_k$ and since by (9.4.3) $q - n_k \le \varepsilon n_k$, we therefore obtain

$$\begin{aligned} &f^q\big(B(x, Cr(\varepsilon)\exp-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)\big) \subset \\ &\quad\subset B(f^q(x), CK(\varepsilon)r(\varepsilon)e^{-\chi_\mu(n-n_k)}\exp(\varepsilon(n_k - (2+\log\|f'\|)n))\exp((q-n_k)\log\|f'\|) \\ &\quad\subset B(f^q(x), CK(\varepsilon)r(\varepsilon)\exp\big(\varepsilon(n_k\log\|f'\| + n_k - 2n - n\log\|f'\|)\big) \\ &\quad\subset B(f^q(x), CK(\varepsilon)r(\varepsilon)e^{-\varepsilon n}) \subset B(f^q(x), e^{-\varepsilon q}). \end{aligned}$$

Combining this, (9.4.2), and (9.4.4), we get

$$B\big(x, Cr(\varepsilon)\exp-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)\big) \subset \bigvee_{j=0}^{n} f^{-j}(\mathcal{P})(x).$$

Therefore, applying Theorem 1.5.5 (the Shanon-McMillan-Breiman Theorem), we have

$$\liminf_{n\to\infty} -\frac{1}{n}\log\mu\big(B(x, Cr(\varepsilon)\exp-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)\big) \ge \mathrm{h}_\mu(f,\mathcal{P}) \ge \mathrm{h}_\mu(f) - \varepsilon$$

It means that denoting the number $Cr(\varepsilon)\exp-(\chi_\mu + (2+\log\|f'\|)\varepsilon)n)$ by r_n, we have

$$\liminf_{n\to\infty}\frac{\log\mu(B(x,r_n))}{\log r_n} \ge \frac{\mathrm{h}_\mu(f)-\varepsilon}{\chi_\mu(f) + (2+\log\|f'\|)\varepsilon}$$

Now, since $\{r_n\}$ is a geometric sequence and since $\varepsilon > 0$ can be taken arbitrarily small, we conclude that for μ-a.e. $x \in X$

$$\liminf_{n\to\infty}\frac{\log\mu(B(x,r)}{\log r} \ge \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)}$$

This completes the proof of (9.4.1). ♣

Remark. Since here $X \subset \mathbb{C}$, we could have considered a partition $\mathcal{P}$ of a neighbourhood of X in $\mathbb{C}$ where $\partial_{\mathcal{P},a}$ would have a more standard sense, see Remark after Corollary 7.1.8.

Now let us prove that

(8.4.5)
$$\limsup_{r\to 0}\frac{\log(\mu(B(x,r)))}{\log r} \le \mathrm{h}_\mu(f)/\chi_\mu(f)$$

for μ-a.e. $x \in X$.

In order to prove this formula we again work in the natural extension $(\tilde{X}, \tilde{f}, \tilde{\mu})$ and we apply Pesin theory. In particular the sets $\tilde{X}(\varepsilon)$, $\tilde{F}(\varepsilon) \subset \tilde{X}(\varepsilon)$ and the radius $r(\varepsilon)$, produced in Corollary 9.2.4 have the same neaning as in the proof of (9.4.1). To begin with notice that there exist two numbers $R > 0$ and $0 < Q < \min\{1, r(\varepsilon)/2\}$ such that the foloowing two conditions are satisfied.

(9.4.6) If $z \notin B(\mathrm{Crit}(f), R)$, then $f|_{B(z,Q)}$ is injective.

(9.4.7) If $z \in B(\mathrm{Crit}(f), R)$, then $f|_{B(z,Q\mathrm{dist}(z,\mathrm{Crit}(f)))}$ is injective.

Observe also that if z is sufficiently close to a critical point c, then $f'(z)$ is of order $(z-c)^{q-1}$, where $q \geq 2$ is the order of critical point c. In particular the quotient of $f'(z)$ and $(z-c)^{q-1}$ remains bounded away from 0 and ∞ and therefore there exists a constant number $B > 1$ such that $|f'(z)| \leq B\mathrm{dist}(z, \mathrm{Crit}(f))$. So, in view of Theorem 9.2.2, the logarithm of the function $\rho(z) = Q\min\{1, \mathrm{dist}(z, Crit(f))$ is integrable and consequently Lemma 9.3.2 applies. Let $\mathcal{P}$ be the Mañé's partition produced by this lemma. Then $B(x, \rho(x)) \supset \mathcal{P}(x)$ for μ-a.e. $x \in X$, say for a subset X_ρ of X of measure 1. Consequently

$$B_n(x,\rho) = \bigcap_{j=0}^{n-1} f^{-j}\big(B(f^j(x), \rho(f^j(x)))\big) \supset \mathcal{P}_0^n(x) \tag{9.4.8}$$

for every $n \geq 1$ and every $x \in X_\rho$. By our choice of Q and the definition of ρ, the function f is injective on all balls $B(f^j(x), \rho(f^j(x)))$, $j \geq 0$, and therefore f^k is injective on the set $B_n(x,\rho)$ for every $0 \leq k \leq n-1$. Now, let $x \in F(\varepsilon) \cap X_\rho$ and let k be the greatest subscript such that $q = n_k(x) \leq n-1$. Denote by f_x^{-q} the unique holomorphic inverse branch of f^q produced by Corollary 9.2.4 which sends $f^q(x)$ to x. Clearly $B_n(x,\rho) \subset f^{-q}(B(f^q(x), \rho(f^q(x))))$ and since f^q is injective on $B_n(x,\rho)$ we even have

$$B_n(x,\rho) \subset f_x^{-q}(B(f^q(x), \rho(f^q(x)))).$$

By Corollary 9.2.4 $\mathrm{diam}\big(f_x^{-q}(B(f^q(x), \rho(f^q(x))))\big) \leq K\exp(-q(\chi_\mu - \varepsilon))$. Since by (9.4.3), $n \leq q(1+\varepsilon)$ we finally deduce that

$$B_n(x,\rho) \subset B\big(x, K\exp(-n\frac{\chi_\mu - \varepsilon}{1+\varepsilon})\big).$$

Thus, in view of (9.4.8)

$$B\left(x, K\exp(-n\frac{\chi_\mu - \varepsilon}{1+\varepsilon})\right) \supset \mathcal{P}_0^n(x).$$

Therefore, denoting by r_n the radius of the ball above, it follows from Shanon-McMillan-Breiman theorem that for μ-a.e $x \in X$

$$\limsup_{n\to\infty} -\frac{1}{n}\log\mu(B(x, r_n) \leq \mathrm{h}_\mu(f, \mathcal{P}) \leq \mathrm{h}_\mu(f).$$

So

$$\limsup_{n\to\infty} \frac{\log \mu(B(x,r_n))}{\log r_n} \le \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)-\varepsilon}(1+\varepsilon).$$

Now, since $\{r_n\}$ is a geometric sequence and since ε can be taken arbitrarily small, we conclude that for μ-a.e. $x \in X$

$$\limsup_{n\to\infty} \frac{\log \mu(B(x,r))}{\log r} \le \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)}.$$

This completes the proof of (9.4.5) and because of (9.4.1) also the proof of Theorem 9.4.2. ♣

§9.5 PRESSURE-LIKE DEFINITION OF THE FUNCTIONAL $\mathrm{h}_\mu + \int \phi\, d\mu$.

In this section we prepare some general tools used in the next section to approximate topological pressure on hyperbolic sets. No smoothness is assumed here, we work in purely metric setting only. Our exposition is similar to that contained in Chapter 2.

Let $T : X \to X$ be a continuous map of a compact metric space (X, ρ) and let μ be a Borel probability measure on X. Given $\varepsilon > 0$ and $0 \le \delta \le 1$ a set $E \subset X$ is said to be $\mu - (n, \varepsilon, \delta)$-*spanning* if

$$\mu\Big(\bigcup_{x\in E} B_n(x,\varepsilon)\Big) \ge 1-\delta.$$

Let $\phi : X \to I\!R$ be a continuous function. We define

$$Q_\mu(T,\phi,n,\varepsilon,\delta) = \inf_E \Big\{\sum_{x\in E} \exp S_n\phi(x)\Big\}$$

where the infimum is taken over all $\mu - (n, \varepsilon, \delta)$-spanning sets E. The main result of this section is the following.

Theorm 9.5.1. For every $0 < \delta < 1$ and every ergodic measure μ

$$\mathrm{h}_\mu(T) + \int \phi\, d\mu = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log Q_\mu(T,\phi,n,\varepsilon,\delta) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q_\mu(T,\phi,n,\varepsilon,\delta)$$

Proof. Denote the the number following the first equality sign by $\underline{\mathrm{P}}_\mu(T,\phi,\delta)$ and the number following the second equality sign by $\overline{\mathrm{P}}_\mu(T,\phi,\delta)$. First, following essentially the proof of the Part I of Theorem 2.3.1, we shall show that

$$\underline{\mathrm{P}}_\mu(T,\phi,\delta) \ge \mathrm{h}_\mu(T) + \int \phi\, d\mu \tag{9.5.1}$$

Indeed, similarly as in that proof consider a finite partition $\mathcal{U} = \{A_1, \dots, A_s\}$ of X into Borel sets and compact sets $B_i \subset A_i$, $i = 1, 2, \dots, A_s\}$, such that for the partition $\mathcal{V} = \{B_1, \dots, B_s, X \setminus (B_1 \cup \ldots \cup B_s)\}$ we have $H_\mu(\mathcal{U}|\mathcal{V}) \le 1$. For every $\theta > 0$ and $q \ge 1$, set

$$X_q = \{x \in X : -\frac{1}{n} \log \mu\big(\mathcal{V}^n(x)\big) \ge \mathrm{h}_\mu(T, \mathcal{V}) - \theta \ \text{ for all } \ n \ge q \text{ and}$$
$$\frac{1}{n} S_n\phi(x) \ge \int \phi \, d\mu - \theta \ \text{ for all } \ n \ge q\}$$

Fix now $0 \le \delta < 1$. It follows from Shannon-McMillan-Breiman theorem and Birkhoff's ergodic theorem that for q large enough $\mu(X_q) > \delta$. Take $0 < \varepsilon < \frac{1}{2} \min\{\rho(B_i, B_j) : 1 \le i < j \le s\} > 0$ so small that

$$|\phi(x) - \phi(y)| < \theta$$

if $\rho(x, y) \le \varepsilon$. Since for every $x \in X$ the set $B_n(x, \varepsilon) \cap X_q$ can be covered by at most 2^n elements of $\mathcal{V}^n$,

$$\mu(B_n(x, \varepsilon) \cap X_q) \le \exp\big(n(\log 2 - \mathrm{h}_\mu(T, \mathcal{V}) + \theta)\big).$$

Now let E be a $\mu - (n, \varepsilon, \delta)$-spanning set for $n \ge q$, and consider the set $E' = \{x \in E : B_n(x, \varepsilon) \cap X_q \ne \emptyset\}$. Take any point $y(x) \in B_n(x, \varepsilon) \cap X_q$. Then by the choice of ε, $S_n\phi(x) - S_n\phi(y) > -n\theta$. Therefore we have

$$\begin{aligned}
\sum_{x \in E} &\exp S_n\phi(x) \exp\big(-n\big(\mathrm{h}_\mu(T, \mathcal{V}) + \int \phi \, d\mu - 3\theta - \log 2\big)\big) \ge \\
&\ge \sum_{x \in E'} \exp S_n\phi(x) \exp\big(-n\big(\mathrm{h}_\mu(T, \mathcal{V}) + \int \phi \, d\mu - 3\theta - \log 2\big)\big) \\
&= \sum_{x \in E'} \exp\big(S_n\phi(x) - n \int \phi \, d\mu\big) \exp\big(-n(\mathrm{h}_\mu(T, \mathcal{V}) - 3\theta - \log 2)\big) \\
&= \sum_{x \in E'} \exp\big(S_n\phi(x) - S_n\phi(y) + S_n\phi(y) - n \int \phi \, d\mu\big) \exp\big(-n\big(\mathrm{h}_\mu(T, \mathcal{V}) - 3\theta - \log 2\big)\big) \\
&\ge \sum_{x \in E'} \exp(-n\theta) \exp(-n\theta) \exp(2n\theta) \exp\big(-n(\mathrm{h}_\mu(T, \mathcal{V}) - \theta - \log 2)\big) \\
&= \sum_{x \in E'} \exp\big(n(\log 2 - \mathrm{h}_\mu(T, \mathcal{V}) + \theta)\big) \\
&\ge \sum_{x \in E'} \mu(B_n(x, \varepsilon) \cap X_q) \ge \mu(X_q) - \delta > 0
\end{aligned}$$

which implies that

$$Q_\mu(T, \phi, n, \varepsilon, \delta) \ge \mathrm{h}_\mu(T, \mathcal{V}) + \int \phi \, d\mu - 3\theta - \log 2.$$

Since $\theta > 0$ is an arbitrary number and since $\mathrm{h}_\mu(T, \mathcal{U}) \le \mathrm{h}_\mu(T, \mathcal{V}) + H_\mu(\mathcal{U}|\mathcal{V}) \le \mathrm{h}_\mu(T, \mathcal{V}) + 1$, letting $\varepsilon \to 0$, we get

$$\underline{\mathrm{P}}_\mu(T, \phi, \delta) \ge \mathrm{h}_\mu(T, \mathcal{U}) - 1 + \int \phi \, d\mu - \log 2$$

Therefore, by the definition of entropy of an automorphism, $\underline{\mathrm{P}}_\mu(T,\phi,\delta) \ge \mathrm{h}_\mu(T) + \int \phi\, d\mu - \log 2 - 1$. Using now the standard trick, actually always applied in the setting we are whose point is to replace T by its arbitrary iterates T^k and ϕ by $S_k\phi$, we obtain $k\underline{\mathrm{P}}_\mu(T,\phi,\delta) \ge k\mathrm{h}_\mu(T) + k\int \phi\, d\mu - \log 2 - 1$. So, dividing this inequality by k, and letting $k \to \infty$, we finally obtain

$$\underline{\mathrm{P}}_\mu(T,\phi,\delta) \ge \mathrm{h}_\mu(T) + \int \phi\, d\mu$$

Now let us prove that

$$\overline{\mathrm{P}}_\mu(T,\phi,\delta) \le \mathrm{h}_\mu(T) + \int \phi\, d\mu \tag{9.5.2}$$

where $\overline{\mathrm{P}}_\mu(T,\phi,\delta)$ denotes limsup appearing in the statement of Theorm 9.5.1. Indeed, fix $0 < \delta < 1$, then $\varepsilon > 0$ and $\theta > 0$. Let P be a finite partition of X of diameter $\le \varepsilon$. By Shannon-McMillan-Breiman theorem and Birkhoff's ergodic theorem there exists a Borel set $Z \subset X$ such that $\mu(Z) > 1 - \delta$ and

$$\frac{1}{n}S_n\phi(x) \le \int \phi\, d\mu + \theta \quad \text{and} \quad -\frac{1}{n}\log \mu(\mathrm{P}^n(x)) \le \mathrm{h}_\mu(T) + \theta \tag{9.5.3}$$

for every n large enough and all $x \in Z$. From each element of P^n having non-empty intersection with Z choose one point obtaining, say, a set $\{x_1, x_2, \dots, x_q\}$. Then $B_n(x_j,\varepsilon) \supset \mathrm{P}^n(x_j)$ for every $j = 1, 2, \dots, q$ and therefore the set $\{x_1, x_2, \dots, x_q\}$ is $\mu - (n,\varepsilon,\delta)$-spanning. By the second part of (9.5.3) we have $q \le \exp(n(\mathrm{h}_\mu(T) + \theta))$. Using also the first part of (9.5.3), we get

$$\sum_{j=1}^{q} \exp S_n\phi(x_j) \le \exp(n(\mathrm{h}_\mu(T) + \theta + \int \phi\, d\mu + \theta))$$

Therefore $Q_\mu(T,\phi,n,\varepsilon,\delta) \le \exp(n(\mathrm{h}_\mu(T) + \theta + \int \phi\, d\mu + \theta))$ and letting consequtively $n \to \infty$ and $\varepsilon \to 0$, we obtain $\overline{\mathrm{P}}_\mu(T,\phi,\delta) \le \mathrm{h}_\mu(T) + \int \phi\, d\mu + 2\theta$. Since θ is an arbitrary positive number, (9.5.2) is proved. This and (9.5.1) complete the proof of Theorem 9.5.1. ♣.

§9.6 KATOK'S THEORY – HYPERBOLIC SETS, PERIODIC POINTS, AND PRESSURE

In this section we again come back to the setting of Section 9.1. So, let X be a compact subset of the closed complex plane $\overline{\mathbb{C}}$ and let $f : X \to X$ be a continuous map that can be analytically extended to an open neighbourhood $U(f)$ of X.

Let μ be an f-invariant ergodic measure on X with positive Lyapunov exponent. $\mathrm{h}_\mu(f)$ and let $\phi : X \to \mathbb{R}$ be a real continuous function. Our first aim is to show that the number

$h_\mu(f)+\int \phi\, d\mu$ can be approximated by the topological pressures of ϕ on hyperbolic subsets of X and then as a straightforward consequence we will obtain the same approximation for the topological pressure $P(f,\phi)$.

Theorm 9.6.1. If μ is an f-invariant ergodic measure on X with positive Lyapunov expenent χ_μ and if $\phi : X \to I\!R$ is a real-valued continuous function, then there exists a sequence X_k, $k = 1, 2, \dots$, of compact f- invariant subsets of U such that for every k the restriction $f|_{X_k}$ is a conformal expanding repeller,

$$\liminf_{k\to\infty} P(f|_{X_k}, \phi) \geq h_\mu(f) + \int \phi\, d\mu$$

and if μ_k is any ergodic f-invariant measure on X_k, then the sequence μ_k, $k = 1, 2, \dots$, converges to μ in the weak-*-topology on a closed neighbourhood of X.

Proof. Since $P(f|_{X_k}, \phi + c) = P(f|_{X_k}, \phi) + c$ and since $h_\mu(f) + \int(\phi + c)\, d\mu = h_\mu(f) + \int \phi\, d\mu + c$, adding a constant if necessary, we can assume that ϕ is positive, that is that $\inf \phi > 0$. As in Section 9.2 we work in the natural extension $(\tilde X, \tilde f, \tilde\mu)$. Given $\delta > 0$ let $\tilde X(\delta)$ and $r(\delta)$ be produced by Corollary 9.2.4. The set $\pi(\tilde X(\delta))$ is assumed to be compact. This corollary implies the existence of a constant $\chi' > 0$ (possibly with a smaller radius $r(\delta)$) such that

$$\text{(9.6.1)} \qquad \operatorname{diam}\bigl(f^{-n}_{x_n}(B(\pi(\tilde x), r(\delta))\bigr) \leq e^{-n\chi'}$$

for all $\tilde x \in \tilde X(\delta)$ and $n \geq 0$. Fix a countable basis $\{\psi_j\}_{j=1}^\infty$ of the Banach space $C(X)$ of all continuous real-valued functions $C(X)$. Fix $\theta > 0$ and an integer $s \geq 1$. In view of Theorem 9.5.1 and continuity of functions ϕ and ψ_i there exists $\varepsilon > 0$ so small that

$$\text{(9.6.2)} \qquad \liminf_{n\to\infty} \frac{1}{n} \log Q_\mu(T, \phi, n, \varepsilon, \delta) - (h_\mu(f) + \int \phi\, d\mu) > -\theta,$$

if $|x - y| < \varepsilon$, then

$$\text{(9.6.3)} \qquad |\phi(x) - \phi(y)| < \theta$$

and

$$\text{(9.6.4)} \qquad |\psi_i(x) - \psi_i(y)| < \frac{1}{2}\theta$$

for all $i = 1, 2, \dots, s$.

Set $\beta = r(\delta)/2$ and fix a finite $\beta/2$-spanning set of $\pi(\tilde X(\delta))$, say $\{x_1, \dots, x_t\}$. That is $B(x_1, \beta/2) \cup \ldots \cup B(x_t, \beta/2) \supset \pi(\tilde X(\delta/2))$. Let $\mathcal{U}$ be a finite partion of X with diameter $< \beta/2$ and let n_1 be sufficiently large that

$$\text{(9.6.5)} \qquad \exp(-n_1\chi') < \min\{\beta/3, K^{-1}\}.$$

Given $n \geq 1$ define

$$\begin{aligned}\tilde{X}_{n,s} = \{\tilde{x} \in \tilde{X}(\delta) : \tilde{f}^q(\tilde{x}) \in \tilde{X}(\delta) \ \text{ and } \ &\pi(\tilde{f}^q(\tilde{x})) \in \mathcal{U}(\tilde{x}) \\ &\text{for some } \ q \in [n+1, (1+\theta)n] \\ &\text{and } \left| \frac{1}{k} S_k(\psi_i)(\pi(\tilde{x})) - \int \psi_i \, d\mu \right| < \frac{1}{2}\theta \\ &\text{for every } k \geq n \ \text{ and all } \ i = 1, 2, \ldots, s\}.\end{aligned}$$

By Birkhoff's ergodic theorem $\lim_{n\to\infty} \mu(\tilde{X}_{n,s}) = \mu(\tilde{X}(\delta)) > 1 - \delta$. Therefore there exists $n \geq n_1$ so large that $\mu(\tilde{X}_{n,s}) > 1 - \delta$. Let $X_{n,s} = \pi((\tilde{X}_{n,s}))$. Then $\mu(X_{n,s}) > 1 - \delta$ and let $E_n \subset X_{n,s}$ be a maximal (n, ε)-separated subset of $X_{n,s}$. Then E_n is a spanning set of $X_{n,s}$ and therefore it follows from (8.6.2) that for all n large enough

$$\frac{1}{n} \log \sum_{x \in E_n} \exp S_n\phi(x) - (\mathrm{h}_\mu(f) + \int \phi \, d\mu) > -\theta.$$

Equivalently

$$\sum_{x \in E_n} \exp(S_n\phi(x)) > \exp(n(\mathrm{h}_\mu(f) + \int \phi \, d\mu - \theta)).$$

For every $q \in [n+1, (1+\theta)n]$ let

$$V_q = \{x \in E_n : f^q(x) \in \mathcal{U}(x)\}$$

and let $m = m(n)$ be a value of q that maximizes $\sum_{x \in V_q} \exp(S_n\phi(x))$. Since $\bigcup_{q=n+1}^{(1+\theta)n} V_q = E_n$, we thus obtain

$$\begin{aligned}\sum_{x \in V_m} \exp S_n\phi(x) &\geq (n\theta)^{-1} \sum_{q=n+1}^{(1+\theta)n} \sum_{x \in V_q} \exp S_n\phi(x) \\ &\geq (n\theta)^{-1} \sum_{x \in E_n} \exp(S_n\phi(x)) \geq \exp(n(\mathrm{h}_\mu(f) + \int \phi \, d\mu - 2\theta)).\end{aligned}$$

Consider now the sets $V_m \cap B(x_j, \beta/2)$, $1 \leq j \leq t$ and choose the value $i = i(m)$ of j that maximizes $\sum_{x \in V_m \cap B(x_j, \beta/2)} \exp(S_n\phi(x))$. Thus, writing D_m for $V_m \cap B(x_{i(m)}, \beta/2)$ we have $V_m = \bigcup_{j=1}^t V_m \cap B(x_i, \beta/2)$ and

$$\sum_{x \in D_m} \exp S_n\phi(x) \geq \frac{1}{t} \exp(n(\mathrm{h}_\mu(f) + \int \phi \, d\mu - 2\theta)).$$

Since ϕ is positive, this implies that

$$\sum_{x \in D_m} \exp S_m\phi(x) \geq \frac{1}{t} \exp(n(\mathrm{h}_\mu(f) + \int \phi \, d\mu - 2\theta)). \tag{9.6.6}$$

Now, if $x \in D_m$, then $|f^m(x) - x_i| \le |f^m(x) - x| + |x - x_i| < \beta/2 + \beta/2 = \beta$ and therefore

$$f^m(x) \in B(x_i, \beta) \subset B(f^m(x), 2\beta).$$

Thus, by (9.6.1) and as $m \ge n \ge n_1$, we have $\mathrm{diam}\big(f_x{-}m(B(f^m(x), 2\beta)\big) \le \exp(-m\chi') < \beta/3$, where $\tilde{x} \in \pi^{-1}(x) \cap \tilde{X}_{n,s}$. Therefore

$$f_x^{-m}(B(x_i, \beta)) \subset B\left(x_i, \frac{\beta}{2} + \frac{\beta}{3}\right) = B\left(x_i, \frac{5}{6}\beta\right)$$

In particular

$$\overline{f_x^{-m}(B(x_i, \beta))} \subset B(x_i, \beta) \tag{9.6.7}$$

Consider now two distinct points $y_1, y_2 \in D_m$. Then $f_{y_2}^{-m}(B(x_i, \beta)) \cap f_{y_1}^{-m}(B(x_i, \beta)) = \emptyset$ and decreasing β a little bit, if necessary, we may assume that

$$f_{y_2}^{-m}(\overline{B(x_i, \beta)}) \cap f_{y_1}^{-m}(\overline{B(x_i, \beta)}) = \emptyset.$$

Let

$$\xi = \min\left\{\beta, \min\left\{\mathrm{dist}\big(f_{y_2}^{-m}(\overline{B(x_i, \beta)}), f_{y_1}^{-m}(\overline{B(x_i, \beta)})\big) : y_1, y_2 \in D_m, y_1 \ne y_2\right\}\right\}.$$

Define now inductively the sequence of sets $\{X^{(j)}\}_{j=0}^{\infty}$ contained in $U(f)$ by setting

$$X^{(0)} = \overline{(B(x_i, \beta)} \quad \text{and} \quad X^{(j+1)} = \bigcup_{x \in D_m} f_{x_0}^{-m}(X^{(j)})$$

By (9.6.7),$\{X^{(j)}\}_{j=0}^{\infty}$, is a descending sequence of non-empty compact sets, and therefore the intersection

$$X^* = X^*(\theta, s) = \bigcap_{j=0}^{\infty} X^{(j)}$$

is also a non-empty compact set. Moreover, by the construction $f^m(X^*) = X^*$, $f^m|_{X^*}$ is topologically conjugate to the full one-sided shift generated by an alphabet consisting of $\#D_m$ elements and it immediately follows from Corollary 9.2.4 that $f^m|_{X^*}$ is an expanding map. Since $f^m|_{X^*}$ is an open map, it is straightforward to check that the triple (f^m, X^*, U_m) is a conformal expanding repeller with a sufficiently small neighborhood U_m of X^*. Thus $(f, X(\theta, s), W_s)$, is also a conformal expanding repeller, where

$$X(\theta, s) = \bigcup_{l=0}^{m-1} f^l(X^*) \quad \text{and} \quad W_s = \bigcup_{l=0}^{m-1} f^l(U_m)$$

Fix now an integer $j \geq 1$. For any j-tuple $(z_0, z_1, \ldots, z_{j-1})$, $z_l \in D_m$ choose exactly one point y from the set $f_{z_{j-1}}^{-m} \circ f_{z_{j-2}}^{-m} \circ \ldots \circ f_{z_0}^{-m}(X^*)$ and denote the made up set by A_j. Since by (9.6.3) and (9.6.5) $S_{jm}\phi(y) \geq \sum_{l=0}^{j-1} S_m\phi(z_l) - jm\theta$ we see that

$$\sum_{y \in A_j} \exp S_{jm}\phi(y) \geq \Big(\sum_{x \in D_m} \exp S_m\phi(x)\Big)^j \exp(-jm\theta)$$

and

$$\frac{1}{j} \log \sum_{y \in A_j} \exp S_{jm}\phi(y) \geq \log \sum_{x \in D_m} \exp S_m\phi(x) - m\theta$$

In view of the definition of ξ, the set A_j is (j, ξ)-separated for f^m and ξ is an expansive constant for f^m. Hence, letting $j \to \infty$ we obtain

$$\begin{aligned} \mathrm{P}(f^m|_{X^*}, S_m\phi) &\geq \log \sum_{x \in D_m} \exp S_m\phi(x) - m\theta \\ &\geq n\big(\mathrm{h}_\mu(f) + \int \phi \, d\mu - 2\theta\big) - \log t - m\theta \end{aligned}$$

where the last inequality was written in view of (9.6.6). Since $n + 1 \leq m \leq n(1 + \theta)$ and since $\inf \phi > 0$ (and consequently $\mathrm{h}_\mu(f) + \int \phi \, d\mu > 0$), we get

$$\begin{aligned} \mathrm{P}(f|_{X(\theta,s)}, \phi) &= \frac{1}{m}\mathrm{P}(f^m|_{X(\theta,s)}, \phi) \geq \frac{1}{m}\mathrm{P}(f^m|_{X^*}, S_m\phi) \\ &\geq \frac{1}{1+\theta}\left(\mathrm{h}_\mu(f) + \int \phi \, d\mu - 2\theta\right) - \frac{\log t}{m} - \theta \end{aligned}$$

Supposing now that n (and consequently also m) was choosen sufficiently large we get

$$\mathrm{P}(f|_X(\theta, s), \phi) \geq \frac{1}{1+\theta}(\mathrm{h}_\mu(f) + \int \phi \, d\mu) - 4\theta.$$

If now ν is any ergodic f-invariant measure on $X(\theta, s)$, then it follows from the definition of the set $\tilde{X}_{n,s}$, the construction of the set $X(\theta, s)$ and the Birkhoff ergodic theorem that $|\int \psi_i \, d\nu - \int \psi_i \, d\mu| < \theta$ for every $i = 1, 2, \ldots, s$. Therefore putting for example $X_k = X(1/k, k)$, completes the proof of Theorem 9.6.1. ♣

Remark 9.6.2. If the set X is repelling, that is if $\bigcap_{n \geq 0} f^{-n}(U) = X$, then the sets X_k constucted in the proof of Theorem 9.6.1 are all contained in X. In particular we get the following.

Corollary 9.6.3. If the set X is repelling and if $\mathrm{P}(f, \phi) > \sup \phi$, then there exists a sequence X_k, $k = 1, 2, \ldots$, of compact f-invariant subsets of X such that for every k, $f|_{X_k}$ is a conformal expanding repeller,

$$\lim_{k \to \infty} \mathrm{P}(f|_{X_k}, \phi) = \mathrm{P}(f, \phi)$$

and if μ_k is any ergodic f-invariant measure on X_k, then the sequence μ_k, $k = 1, 2, \ldots$, converges to μ in the weak-*-topology on X.

Remark 9.6.4. Of course in Corollary 9.6.3 was sufficient to assume that $\mathrm{P}(f, \phi) = \sup\{\mathrm{h}_\mu(f) + \int \phi \, d\mu\}$ where the supremum is taken over all ergodic invariant measures of positive entropy, which is assured for eaxample by inequality $\mathrm{P}(f, \phi) > \sup \phi$. Besides, if the function ϕ has an equilibrium state of positive entropy, then the sequence μ_k can be choosen to converge to this equilibrium state.

Our last immediate conclusion concerns periodic points.

Corollary 9.6.5. If $f : X \to X$ is repelling and $\mathrm{h}_{\mathrm{top}}(f) > 0$, then f has infinitely many periodic points. Moreover the number of periodic points of period n grows exponentialy fast with n.