SOLUTION FOR APRIL 2023

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Let n be a positive integer. Prove that:

$$(\sqrt{2}-1)^n = \sqrt{m} - \sqrt{m-1} \text{ for some positive integer } m.$$

Solution: If n is even then $m = b_{2n}^2$ and if n is odd then $m = 2a_{2n+1}^2$ where a_n, b_n are defined below.

Proof: Observe that:

$$(\sqrt{2}-1)^n = (-1)^{n+1}(a_n\sqrt{2} - b_n) \tag{1}$$

where a_n and b_n are integers and that $a_1 = 1, b_1 = 1$. Further:

$$(\sqrt{2}-1)^{n+1} = (\sqrt{2}-1)^n(\sqrt{2}-1)$$

$$= (-1)^{n+1}(a_n\sqrt{2} - b_n)(\sqrt{2}-1) = (-1)^{n+2}\left(\sqrt{2}(a_n + b_n) - (2a_n + b_n)\right)$$

and thus:

$$a_{n+1} = a_n + b_n$$

$$b_{n+1} = 2a_n + b_n.$$

It follows from this that $0 < a_n < a_{n+1}$ and $0 < b_n < b_{n+1}$ and it also follows from this that a_n and b_n are integers. In addition, a straightforward computation shows:

$$2a_{n+1}^2 - b_{n+1}^2 = (-1)(2a_n^2 - b_n)$$

and thus it follows by induction that:

$$2a_n^2 - b_n^2 = (-1)^{n+1}. \tag{2}$$

Case 1: n even

Then we see using (1):

$$(\sqrt{2}-1)^{2n} = b_{2n} - a_{2n}\sqrt{2} = \sqrt{b_{2n}^2} - a_{2n}\sqrt{2}.$$

So we will finish this case if we can show

$$a_{2n}\sqrt{2} = \sqrt{b_{2n}^2 - 1}$$

but this follows from (2). This completes Case 1 with $m = b_{2n}^2$.

Case 2: n odd

Then we see using (1):

$$(\sqrt{2}-1)^{2n+1} = a_{2n+1}\sqrt{2} - b_{2n+1} = \sqrt{2a_{2n+1}^2} - b_{2n+1}.$$

So we will finish this case if we can show

$$b_{2n+1} = \sqrt{2a_{2n+1}^2 - 1}$$

but this again follows from (2). This completes Case 2 with $m = 2a_{2n+1}^2$. $\square$