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DIFFERENTIAL OPERATORS FOR HERMITIAN JACOBI FORMS AND HERMITIAN MODULAR FORMS

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Abstract. Kim [14] constructs multilinear differential operators for Hermitian Jacobi forms and Hermitian modular forms. However, her work relies on incorrect actions of differential operators on spaces of Hermitian Jacobi forms and Hermitian modular forms. In particular, her results are incorrect if the underlying field is the Gaussian number field. We consider more general spaces of Hermitian Jacobi forms and Hermitian modular forms over $\mathbb{Q}(i)$, which allow us to correct the corresponding results in [14].

1. Introduction

Differential operators are important tools in the theory of automorphic forms. Rankin-Cohen brackets are certain bilinear differential operators $[\,,]_\nu$ on the space of holomorphic functions on the upper half-plane, indexed by a non-negative integer ν, which take modular forms to modular forms. If f and g are modular forms of weights k and ℓ respectively, then

$$[f,g]_\nu := \sum_{\substack{r,s\geq 0\\ r+s=\nu}} (-1)^r \binom{k+\nu-1}{s}\binom{\ell+\nu-1}{r} \frac{d^r f}{d\tau^r}\frac{d^s g}{d\tau^s}.$$

Moreover, while $[f,g]_0$ is simply the product of the two functions, $[f,g]_1$ satisfies the Jacobi identity, endowing the space of modular forms with a Lie algebra structure. Zagier [18] investigates Rankin-Cohen brackets and their algebraic structure in great detail. For an overview and some of their remarkable applications in different areas of mathematics see, for example [19].

Rankin-Cohen brackets have been constructed for various types of automorphic forms (see for example, [2, 4, 5, 6, 7, 8, 11] among many others). Kim [14] gives a construction of multilinear differential operators (and in particular, Rankin-Cohen brackets) for Hermitian Jacobi forms and Hermitian modular forms of degree 2 over complex quadratic fields K. Unfortunately, her results rely on an incorrect action of a heat operator on Hermitian Jacobi forms and also on an incomplete set of generators for the Hermitian modular group of degree 2, which invalidates her results.

The purpose of this paper is to correct [14] in the case that the underlying field is $K = \mathbb{Q}(i)$. Recently, Richter and the second author [16] introduce a new space of Hermitian Jacobi forms over $\mathbb{Q}(i)$, and they also give the action of the heat operator on this new space. We apply the results in [16] to establish multilinear differential operators (and in particular, Rankin-Cohen brackets) for Hermitian Jacobi forms and Hermitian modular forms correcting the results in [14]. Furthermore, we discuss an explicit example of a Rankin-Cohen bracket of two Hermitian modular forms of degree 2. Throughout this paper we assume that k is an even positive integer.

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2. Differential operators for Hermitian Jacobi forms

In this section we proceed as in [14] to construct differential operators on Hermitian Jacobi forms. We start by briefly recalling the definition of Hermitian Jacobi forms over the Gaussian number field $\mathbb{Q}(i)$ in [16] and the action of the heat operator on such forms. More on Hermitian Jacobi forms may be found in [9]. Throughout, m is a nonnegative integer, $\delta \in \{\pm 1\}$, and $\mathcal{H}$ denotes the usual complex upper half plane. Let $\mathcal{O} := \mathbb{Z}(i)$ be the ring of Gaussian integers with inverse different $\mathcal{O}^{\#} := \frac{1}{2}\mathbb{Z}[i]$, and let $\Gamma(\mathcal{O}) := \{\varepsilon M \,|\, \varepsilon \in \{\pm 1, \pm i\}, M \in \mathrm{SL}_2(\mathbb{Z})\}$ be the Hermitian modular group. For more details on the Hermitian modular group, see for example Chapter 2 of [15].

Definition 2.1. *A holomorphic function $\phi(\tau, z, w) : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ is a Hermitian Jacobi form of weight k, index m, and parity δ, with ε as above, if it satisfies the transformation laws*

- $\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{\varepsilon z}{c\tau+d}, \frac{\varepsilon^{-1}w}{c\tau+d}\Big) = \sigma(\varepsilon)\varepsilon^k\big(c\tau+d\big)^k e^{\frac{2\pi imczw}{c\tau+d}}\phi(\tau, z, w),$
  for every $\varepsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathcal{O})$ *where* $\sigma(\varepsilon) := \begin{cases} 1 & \text{if } \delta = +1 \quad \text{(positive parity)} \\ \varepsilon^2 & \text{if } \delta = -1 \quad \text{(negative parity)}. \end{cases}$
- $\phi\big(\tau, z + \lambda\tau + \mu, w + \overline{\lambda}\tau + \overline{\mu}\big) = e^{-2\pi im\left(\lambda\overline{\lambda}\tau + \overline{\lambda}z + \lambda w\right)}\phi(\tau, z, w)$ *for all* $\lambda, \mu \in \mathcal{O}$.
- *Furthermore, ϕ must have a Fourier expansion of the form*

$$\phi(\tau, z, w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}^{\#} \\ 4mn - |r|^2 \geq 0}} c(n, r)e^{2\pi i(n\tau + rz + \overline{r}w)}.$$

We denote the space of Hermitian Jacobi forms of weight k, index m, and parity δ by $J^{\delta}_{k,m}(\mathcal{O})$. To avoid confusion with other meanings of superscripts, when the parity is known we write $J^{+}_{k,m}$ and $J^{-}_{k,m}$ for positive and negative parity, respectively.

Define the heat operator

$$L_m := \frac{1}{(2\pi i)^2}\Big(8\pi im\frac{\partial}{\partial\tau} - 4\frac{\partial^2}{\partial w \partial z}\Big). \tag{2.1}$$

If $\phi \in J^{\delta}_{k,m}(\mathcal{O})$, then a straightforward computation shows that (for details, see also Lemma 5.1 of [17])

$$L_m(\phi) = \frac{(k-1)m}{3}E_2\phi + \widehat{\phi}, \tag{2.2}$$

where $\widehat{\phi} \in J^{-\delta}_{k+2,m}(\mathcal{O})$ and where E_2 is the usual quasimodular Eisenstein series.

The constructions of multilinear differential operators on Hermitian Jacobi forms in [14] are based on §3 of [12] (see also [5, 7]). In particular, the method in Theorem 3.2 of [14] is valid, and the following theorem deviates only in the action of the heat operator L_m and it's corresponding effect on parity. For brevity we only state key steps and the final results without including the detailed calculations.

Theorem 2.2. *For fixed k, m, and δ, define a formal power series*

$$\widetilde{f}(\tau, z, w; X) = \sum_{\nu=0}^{\infty} \chi_{\nu}(\tau, z, w)X^{\nu}$$

such that $\widetilde{f}$ satisfies the functional equation

$$\widetilde{f}\Big(\frac{a\tau+b}{c\tau+d}, \frac{\varepsilon z}{c\tau+d}, \frac{\varepsilon^{-1}w}{c\tau+d}; \frac{X}{(c\tau+d)^2}\Big) = \sigma(\varepsilon)\varepsilon^k\big(c\tau+d\big)^k e^{\frac{2\pi imczw}{c\tau+d}} e^{\frac{2\pi imcX}{c\tau+d}}\widetilde{f}(\tau,z,w;X) \tag{2.3}$$

for all $\varepsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(\mathcal{O})$.

Furthermore, assume that the Taylor coefficients χ_ν of $\widetilde{f}$ are holomorphic in $\mathcal{H}\times\mathbb{C}^2$ and satisfy

$$\chi_\nu\big(\tau, z+\lambda\tau+\mu, w+\overline{\lambda}\tau+\overline{\mu}\big) = e^{-2\pi im\left(\lambda\overline{\lambda}\tau+\overline{\lambda}z+\lambda w\right)}\chi_\nu(\tau,z,w)$$

for all $\lambda,\mu\in\mathcal{O}$, and have Fourier expansions of the form

$$\chi_\nu(\tau,z,w) = \sum_{n=0}^{\infty}\sum_{\substack{r\in\mathcal{O}^{\#}\\ nm-|r|^2\geq 0}} c_\nu(n,r)e^{2\pi i(n\tau+rz+\overline{r}w)}.$$

Then

$$\xi_\nu := \sum_{\ell=0}^{\nu}\frac{(-1)^\ell L_m^\ell(\chi_{\nu-\ell})(k+2\nu-\ell-3)!}{\ell!(k+2\nu-3)!} \in J_{k+2\nu,m}^{(-1)^\nu\delta}(\mathcal{O}).$$

Proof. The proof is completely analogous to the proof of Theorem 3.2 of [14], and we give only a short sketch. Let $\widetilde{J}_{k,m}^{\delta}$ be the space of formal power series defined above, and let $\widetilde{L}_{k,m} := L_m - (k-1)\frac{\partial}{\partial X} - X\frac{\partial^2}{\partial X^2}$. Then

$$\begin{aligned}
&\widetilde{L}_{k,m}\left(\sigma(\varepsilon)\varepsilon^k(c\tau+d)^{-k}e^{\frac{-2\pi imczw}{c\tau+d}}e^{\frac{-2\pi imcX}{c\tau+d}}\widetilde{f}\left(M\tau, \frac{\varepsilon z}{c\tau+d}, \frac{\varepsilon^{-1}w}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right)\right)\\
&\qquad = \sigma(\varepsilon)\varepsilon^k(c\tau+d)^{-k-2}e^{\frac{-2\pi imczw}{c\tau+d}}e^{\frac{-2\pi imcX}{c\tau+d}}\widetilde{L}_{k,m}\left(\widetilde{f}\right)\left(M\tau, \frac{\varepsilon z}{c\tau+d}, \frac{\varepsilon^{-1}w}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right).
\end{aligned}$$

One finds that $\widetilde{J}_{k,m}^{\delta} \xrightarrow{\widetilde{L}_{k,m}} \widetilde{J}_{k+2,m}^{-\delta} \xrightarrow{\widetilde{L}_{k+2,m}} \widetilde{J}_{k+4,m}^{\delta} \xrightarrow{\cdots} \widetilde{J}_{k+2\nu,m}^{(-1)^\nu\delta}$ and composing with the map $(\tau,z,w,X)\mapsto(\tau,z,w,0)$ yields the claim. $\square$

Note that the Taylor coefficients χ_ν in Theorem 2.2 can be expressed in terms of the Jacobi forms ξ_ν:

$$\chi_\nu(\tau,z,w) = \sum_{\ell=0}^{\nu}\frac{(k+2\nu-2\ell-2)!L_m^\ell(\xi_{\nu-\ell})}{\ell!(k+2\nu-\ell-2)!}.$$

In particular, setting $\xi_0 := f$ for a Hermitian Jacobi form f and $\xi_\nu := 0$ for $\nu\geq 1$ yields the following result.

Theorem 2.3. *Let $f\in J_{k,m}^{\delta}(\mathcal{O})$. Then*

$$\widetilde{f}(\tau,z,w;X) := \sum_{\nu=0}^{\infty}\frac{L_m^\nu(f(\tau,z,w))}{\nu!(k+\nu-2)!}X^\nu$$

satisfies (2.3).

One can apply Theorem 2.3 to define the following multilinear differential operator on Hermitian Jacobi forms, which relies on the action of the heat operator in (2.2).

Theorem 2.4. *Fix $\ell \in \mathbb{N}$. Let $f_i \in J^{\delta_i}_{k_i,m_i}(\mathcal{O})$ for $1 \le i \le \ell$. For any nonnegative integer ν and any $Y = (y_1, \ldots, y_{\ell-1}) \in \mathbb{C}^{\ell-1}$ define*

$$(2.4)\quad [f_1, f_2, \ldots, f_\ell]_{Y,2\nu} :=$$

$$\sum_{\sum_{q=1}^{\ell} r_q + p = \nu} \frac{(-1)^p (k+2\nu-p-3)!}{p!(k+2\nu-3)!} L^p_m \left(\prod_{j=1}^{\ell} \frac{\left(1 - \sum_{i=1}^{j-1} m_i y_i + \sum_{i=j+1}^{\ell} m_i y_j\right)^{r_j}}{r_j!(k_j + r_j - 2)!} L^{r_j}_{m_j}(f_j) \right)$$

where $k := \sum_{i=1}^{\ell} k_i$, $m := \sum_{i=1}^{\ell} m_i$, and $\delta := \prod_{i=1}^{\ell} \delta_i$. Then

$$[f_1, f_2, \ldots, f_\ell]_{Y,2\nu} \in J^{(-1)^\nu \delta}_{k+2\nu,m}(\mathcal{O}).$$

Proof. This proof is completely analagous to Theorem 3.4 of [14], and we again only sketch the proof. Use Theorem 2.3 to define a formal power series in X by

$$F_Y(\tau, z, w; X) := \prod_{q=1}^{\ell} \widetilde{f}_q \left(\tau, z, w; \left(1 - \sum_{i=1}^{q-1} m_i y_i + \sum_{i=q+1}^{\ell} m_i y_q \right) X \right) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau, z, w) X^\nu.$$

Then F_Y satisfies the functional equation (2.3) with $k = \sum_{i=1}^{\ell} k_i$, $m = \sum_{i=1}^{\ell} m_i$, and $\delta = \prod_{i=1}^{\ell} \delta_i$. The result follows by applying Theorem 2.2 to F_Y. $\square$

Remark 2.5. *If $\ell = 2$ and $Y = (0)$ in Theorem 2.4, then (2.4) reduces to the Rankin-Cohen bracket for Hermitian Jacobi forms. That is, for $f \in J^\delta_{k,m}(\mathcal{O})$ and $g \in J^\gamma_{\ell,n}(\mathcal{O})$,*

$$[f, g]_{2\nu} := [f, g]_{Y,2\nu} = \sum_{r+s+p=\nu} \frac{(-1)^p (k+\ell+2\nu-p-3)!}{p!(k+\ell+2\nu-3)!} L^p_{m+n} \left(\frac{L^r_m(f)}{r!(k+r-2)} \frac{L^s_n(g)}{s!(\ell+s-2)} \right).$$

Then $[f, g]_{2\nu} \in J^{(-1)^\nu \delta\gamma}_{k+\ell+2\nu,m+n}(\mathcal{O})$.

3. Differential operators for Hermitian modular forms

Kim §4 [14] constructs a multilinear differential operator on the space of Hermitian modular forms of degree two and weights that are divisible by 4. In this section we follow Kim's construction using Theorem 2.2 and a complete set of generators for $\Gamma_2(\mathcal{O})$. Moreover, we consider also Hermitian modular forms with determinant characters, which allows us to construct a multilinear differential operator on the space of Hermitian modular forms of degree 2 and arbitrary even weights.

We begin by introducing necessary notation. The Hermitian half-space of degree 2 is given by

$$\mathcal{H}_2 := \{ Z \in M_{2\times 2}(\mathbb{C}) \,|\, \tfrac{1}{2i}(Z - \overline{Z}^t) > 0 \},$$

and the Hermitian modular group of degree 2 is defined by

$$\Gamma_2(\mathcal{O}) := \left\{ M \in M_{4\times 4}(\mathcal{O}) \middle| M J \overline{M}^t = J \right\},$$

where $J := \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right)$. As usual, if $Z = \left(\begin{smallmatrix} \tau' & z \\ w & \tau \end{smallmatrix}\right) \in \mathcal{H}_2$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2(\mathcal{O})$, then

$$M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}.$$

Definition 3.1. *Let χ be an abelian character of $\Gamma_2(\mathcal{O})$. A holomorphic function $F : \mathcal{H}_2 \to \mathbb{C}$ is a Hermitian modular form of weight k, degree 2, and character χ if*

$$F(M\langle Z\rangle) = \chi(M)\det(CZ+D)^k F(Z) \tag{3.1}$$

for all $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_2(\mathcal{O})$.

We denote the space of Hermitian modular forms of weight k, degree 2, and character χ by $M_k(\Gamma_2(\mathcal{O}), \chi)$.

Consider a Hermitian modular form $F \in M_k(\Gamma_2(\mathcal{O}), \chi)$. Then writing $Z = \left(\begin{smallmatrix} \tau' & z \\ w & \tau \end{smallmatrix}\right) \in \mathcal{H}_2$ yields the so-called Fourier-Jacobi expansion

$$F(Z) = \sum_{m=0}^{\infty} f_m(\tau, z, w) e^{2\pi i m\tau'},$$

where $f_m \in J^{\delta}_{k,m}(\mathcal{O})$, and δ is determined by the relation

$$f_m^*(Z) := f_m(\tau, z, w)e^{2\pi i m\tau'} = \chi(M)\det(CZ+D)^{-k} f_m^*(M\langle Z\rangle)$$

for all $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_2(\mathcal{O})$.

For our purposes, it is more convenient to use the following set of generators, rather than the more standard generators in Krieg [15]. Recall from Aoki [1] that $\Gamma_2(\mathcal{O})$ is generated by the following matrices:

$$\begin{aligned}
\mathcal{R} &:= \left\{ \left(\begin{smallmatrix} 1 & 0 & a & b+ic \\ 0 & 1 & b-ic & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \,\middle|\, a,b,c,d \in \mathbb{Z} \right\}, \\
\mathcal{S} &:= \left\{ \left(\begin{smallmatrix} 1 & a+ib & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a+ib & 1 \end{smallmatrix}\right) \,\middle|\, a,b \in \mathbb{Z} \right\}, \\
\mathcal{T} &:= \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{smallmatrix}\right) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \right\}, \\
X &:= \left(\begin{smallmatrix} -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \\
Y &:= \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right).
\end{aligned} \tag{3.2}$$

In particular, if $M = X = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)$, then $\det(CZ+D)^k = i^k$, and one finds that Hermitian modular forms with trivial character $\chi = 1$ correspond to Hermitian Jacobi forms of positive parity and if $k \equiv 2 \pmod 4$, then Hermitian modular forms with a determinant character (i.e., $\chi(M) = \det^{-k/2}$ in (3.1)) correspond to Hermitian Jacobi forms of negative parity. This relation follows from the fact by Braun [3] that when $M \in \Gamma_2(\mathcal{O})$, $\det M = \epsilon^2$ for some unit ϵ, which is compatible with Definition 2.1.

We now give the multilinear differential operator for Hermitian modular forms.

Theorem 3.2. *Let $F_i \in M_{k_i}(\Gamma_2(\mathcal{O}), \chi_i)$, where $\chi_i = 1$ or $\chi_i = \det^{-k_i/2}$ ($1 \le i \le \ell$). For a nonnegative integer ν, define*

$$[F_1, \ldots, F_\ell]_\nu := \sum_{\sum_{q=1}^{\ell} r_q + p = \nu} \frac{(-1)^p (k+2\nu-p-3)!}{p!(k+2\nu-3)!} \mathfrak{D}^p \left(\prod_{i=1}^{\ell} \frac{1}{r_i!(k_i+r_i-2)!} \mathfrak{D}^{r_i}(F_i) \right), \tag{3.3}$$

where $\mathfrak{D} := \frac{1}{\pi^2}\left(\frac{\partial^2}{\partial\tau\partial\tau'} - \frac{\partial^2}{\partial z\partial w}\right)$ and $k := \sum_{i=1}^{\ell} k_i$.

Then $[F_1,\dots,F_\ell]_\nu \in M_{k+2\nu}(\Gamma_2(\mathcal{O}),(\chi_1\cdots\chi_\ell)\det^{-\nu})$.

Proof. The Fourier-Jacobi expansions $F_j(Z)=\sum_{m\geq 0} f_m^{(j)}(\tau,z,w)e^{2\pi i m\tau'}$ yield the Fourier-Jacobi expansion of $[F_1,\dots,F_\ell]_\nu$:

$$[F_1,\dots,F_\ell]_\nu = \sum_{j_1\geq 0}\cdots\sum_{j_\ell\geq 0}[f_{j_1}^{(1)},\dots,f_{j_\ell}^{(\ell)}]_{0,2\nu}(\tau,z,w)e^{2\pi i(j_1+\cdots+j_\ell)\tau'}.$$

Consider the generators in (3.2). If $M=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ where $M\in\mathcal{R},\mathcal{S},\mathcal{T}$, or $M=Y$, then Theorem 2.4 implies that

$$[F_1,\dots,F_\ell]_\nu(M\langle Z\rangle)=\det(CZ+D)^{k+2\nu}[F_1,\dots,F_\ell]_\nu(Z).$$

Finally, if $M=X$, then Theorem 2.4 implies that

$$[F_1,\dots,F_\ell]_\nu(X\langle Z\rangle)=\Big(\prod_{i=1}^{\ell}\chi_i(X)\Big)\det(X)^\nu\det(CZ+D)^{k+2\nu}[F_1,\dots,F_\ell]_\nu(Z).$$

The claim follows. $\square$

Remark 3.3. *If* $\ell=2$ *in Theorem 3.2, then* (3.3) *reduces to the Rankin-Cohen bracket for Hermitian modular forms. That is, for* $F\in M_k(\Gamma_2(\mathcal{O}),\chi)$ *and* $G\in M_\ell(\Gamma_2(\mathcal{O}),\psi)$,

$$[F,G]_\nu=\sum_{r+s+p=\nu}\frac{(-1)^p(k+\ell+2\nu-p-3)!}{p!(k+\ell+2\nu-3)!}\mathfrak{D}^p\left(\frac{\mathfrak{D}^r(F)}{r!(k+r-2)!}\frac{\mathfrak{D}^s(G)}{s!(\ell+s-2)!}\right).$$

Then $[F,G]_\nu\in M_{k+\ell+2\nu}(\Gamma_2(\mathcal{O}),\chi\cdot\psi\cdot\det(M)^\nu)$.

We conclude with a concrete example of the first Rankin-Cohen bracket of two Hermitian modular forms.

Example 3.4. *Recall from §2 of* [10] *the Hermitian Eisenstein series* $E_k\in M_k(\Gamma_2(\mathcal{O}),\det^{-\frac{k}{2}})$. *Theorem 3.2 asserts that* $[E_4,E_6]_1\in M_{12}(\Gamma_2(\mathcal{O}),\det^{-6})=M_{12}(\Gamma_2(\mathcal{O}),1)$. *Moreover,* $[E_4,E_6]_1$ *is a so-called symmetric Hermitian modular form, as defined in §2 of* [10].

Graded rings of Hermitian modular forms were determined in [10]. *In particular, Theorem 10b) of* [10] *says that if* $k\equiv 0 \pmod 4$, *then any symmetric* $f\in M_k(\Gamma_2(\mathcal{O}),1)$ *is a polynomial in* E_4, E_6^2, E_6E_{10}, E_{10}^2, E_{12}, *and* $\phi_4^2\in M_8(\Gamma_2(\mathcal{O}),1)$ *where* ϕ_4 *is given in Corollary 4 of* [10]. *A direct computation (using the formula for Fourier coefficients of Hermitian Eisenstein series in* [13]*) reveals that*

$$\begin{aligned}[E_4,E_6]_1 &= \frac{1}{144}\mathfrak{D}(E_4)E_6+\frac{1}{240}E_4\mathfrak{D}(E_6)-\frac{1}{432}\mathfrak{D}(E_4E_6)\\ &= 16E_{12}-\frac{16\cdot 441}{691}E_4^3-\frac{16\cdot 250}{691}E_6^2+\frac{56}{15}E_4\phi_4^2.\end{aligned}$$

Finally, inspecting coefficients shows that $f_{12}:=E_{12}-\frac{441}{691}E_4^3-\frac{250}{691}E_6^2$ *from Corollary 2 of* [10]*and* $E_4\phi_4^2$ *are linearly independent cusp forms of weight* 12 *with trivial character, and one can also write the above bracket as*

$$[E_4,E_6]_1=16f_{12}+\frac{56}{15}E_4\phi_4^2.$$

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