SOLUTION FOR DECEMBER 2023

Let n be a positive integer and let $x > 0$. Define:

$$f(x) = \sum_{k=1}^{2n} \frac{x^{\frac{2k}{2n+1}}}{x^{\frac{2k}{2n+1}} + x}.$$

Determine $f(2023)$.

Solution:

$$f(x) = \sum_{k=1}^{2n} \frac{x^{\frac{2k}{2n+1}}}{x^{\frac{2k}{2n+1}} + x} = n \text{ for all } x > 0$$

and so in particular $f(2023) = n$.

Proof: Let us look at the sum of the $k = 1$ term and the $k = 2n$ term. Then we obtain:

$$\frac{x^{\frac{2}{2n+1}}}{x^{\frac{2}{2n+1}} + x} + \frac{x^{\frac{4n}{2n+1}}}{x^{\frac{4n}{2n+1}} + x}.$$

Notice that if we divide the numerator and denominator of the first term by $x^{\frac{2}{n+1}}$ we obtain:

$$\frac{x^{\frac{2}{2n+1}}}{x^{\frac{2}{2n+1}} + x} = \frac{1}{1 + x^{\frac{2n-1}{2n+1}}}.$$

Next notice that if we divide the numerator and denominator of the second term by x we obtain:

$$\frac{x^{\frac{4n}{2n+1}}}{x^{\frac{4n}{2n+1}} + x} = \frac{x^{\frac{2n-1}{2n+1}}}{1 + x^{\frac{2n-1}{2n+1}}}.$$

Adding these last two equations we obtain:

$$\frac{x^{\frac{2}{2n+1}}}{x^{\frac{2}{2n+1}} + x} + \frac{x^{\frac{4n}{2n+1}}}{x^{\frac{4n}{2n+1}} + x} = 1.$$

Using a similar argument we see that the sum of the $k = 2$ term and the $k = 2n - 1$ is also 1 as is the sum of the $k = 3$ and $k = 2n - 2$ terms, etc., and finally the sum of the $k = n$ term and the $k = n + 1$ term is 1. Thus the entire sum 1 added n times and hence we obtain:

$$\sum_{k=1}^{2n} \frac{x^{\frac{2k}{2n+1}}}{x^{\frac{2k}{2n+1}} + x} = n \text{ for all } x > 0.$$