

SOLUTION FOR JANUARY 2021

Correct solutions were submitted by:

Michalis Paizanis

Find the equation of a circle $(x - h)^2 + (y - k)^2 = r^2$ that has exactly one rational point on it. Also find the equation of a circle that exactly two rational points on it.

SOLUTION: $(0, 0)$ is the only rational point on $(x - \sqrt{2})^2 + y^2 = 2$.

$(0, \pm 1)$ are the only rational points on $(x - \sqrt{2})^2 + y^2 = 3$.

Proof: Suppose $(\frac{m}{n}, \frac{p}{q})$ is a rational point on $(x - \sqrt{2})^2 + y^2 = 2$ with m/n and p/q in lowest terms and $n \neq 0$ and $q \neq 0$. Suppose $m \neq 0$ and $p \neq 0$. Substituting in $(x - \sqrt{2})^2 + y^2 = 2$ gives:

$$2\sqrt{2}q^2mn = q^2m^2 + p^2n^2.$$

Notice the left-hand side is irrational since q, m, n are all nonzero and the right-hand side is rational thus we obtain a contradiction. Also substitution yields that $(0, 0)$ is on $(x - \sqrt{2})^2 + y^2 = 2$. Thus the first statement has been proven.

Similarly, suppose $(\frac{m}{n}, \frac{p}{q})$ is a rational point on $(x - \sqrt{2})^2 + y^2 = 3$ with m/n and p/q in lowest terms and $n \neq 0$ and $q \neq 0$. Suppose $m \neq 0$ and $p \neq 0$. Substituting in $(x - \sqrt{2})^2 + y^2 = 3$ gives:

$$2\sqrt{2}q^2mn = q^2m^2 + (p^2 - 1)n^2.$$

Notice the left-hand side is irrational since q, m, n are all nonzero and the right-hand side is rational thus we obtain a contradiction. Also substitution yields that $(0, \pm 1)$ are on $(x - \sqrt{2})^2 + y^2 = 3$. Thus the second statement has been proven. \square