SOLUTION FOR MARCH 2024

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Determine the values of $p > 0$ for which:

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(m+n)^p}\right)$$

converges and for which $p > 0$ it diverges.

Solution: This converges if and only if $p > 2$.

Proof: Let's first recall the *integral test*. Let $f(x)$ be a positive decreasing continuous function on $[1, \infty)$. Then:

$$\int_2^{\infty} f(x)\,dx \le \sum_{n=2}^{\infty} f(n) \le \int_1^{\infty} f(x)\,dx. \tag{1}$$

It follows from this that $\sum\limits_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Now let $m \ge 1$ and apply the above to $f(x) = \frac{1}{(m+x)^p}$. Then we see from (1) that:

$$g(m) = \int_2^{\infty} \frac{1}{(m+x)^p}\,dx \le \sum_{n=2}^{\infty}\frac{1}{(m+n)^p} \le \int_1^{\infty}\frac{1}{(m+x)^p}\,dx = h(m).$$

Applying the integral test to $g(y)$ and $h(y)$ we see that:

$$\int_2^{\infty}\left(\int_2^{\infty}\frac{1}{(y+x)^p}\,dx\right)dy = \int_2^{\infty} g(y)\,dy \le \sum_{m=2}^{\infty} g(m) \le \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{(m+n)^p}$$

$$\le \sum_{m=1}^{\infty} h(m) \le \int_1^{\infty} h(y)\,dy \le \int_1^{\infty}\left(\int_1^{\infty}\frac{1}{(y+x)^p}\,dx\right)dy.$$

Thus:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n)^p} \text{ converges if and only if } \int_1^{\infty}\int_1^{\infty}\frac{1}{(y+x)^p}\,dx\,dy \text{ converges.}$$

Now notice that:

$$\int_1^{\infty}\frac{1}{(y+x)^p}\,dx \text{ converges if and only if } p > 1.$$

Thus:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n)^p} \text{ diverges if } 0 < p \leq 1.$$

Now assume $p > 1$. Then:

$$\int_1^{\infty}\frac{1}{(y+x)^p}\,dx = \frac{1}{(p-1)}\frac{1}{(y+1)^{p-1}}.$$

Next we see that:

$$\int_1^{\infty}\int_1^{\infty}\frac{1}{(y+x)^p}\,dx\,dy = \int_1^{\infty}\frac{1}{(p-1)}\frac{1}{(y+1)^{p-1}}\,dy$$

and the integral on the right converges if and only if $p - 1 > 1$ i.e. if $p > 2$.

And so finally for $p > 0$ we see:

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(m+n)^p}\right) \text{ converges if and only if } p > 2.$$