SOLUTION FOR MARCH 2023

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Prove:

$$e^{(e^x-1)} = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \tag{1}$$

where $b_0 = 1$ and

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_k. \tag{2}$$

Also prove:

$$b_n = \frac{1}{e} \sum_{n=0}^{\infty} \frac{k^n}{k!}. \tag{3}$$

Remark: Notice that (3) says that the quantity $\frac{1}{e} \sum\limits_{n=0}^{\infty} \frac{k^n}{k!}$ is an integer! (This is not obvious - not to me at least!)

Proof: Let $g(x) = e^{(e^x-1)}$. Next observe $g(0) = 1$ and:

$$g'(x) = e^x g(x)$$

$$g''(x) = (g'(x) + g(x))e^x,$$
$$g'''(x) = (g''(x) + 2g'(x) + g(x))e^x$$
$$g''''(x) = (g'''(x) + 3g''(x) + 3g'(x) + g(x))e^x.$$

Notice that the numbers we obtain as coefficients are exactly the numbers that one obtains in Pascal's triangle. In fact one can show by induction that:

$$g^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(x). \tag{4}$$

Now let us write:

$$g(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

and notice that:

$$b_n = g^{(n)}(0). \tag{5}$$

Then using (4)-(5) we see:

$$b_{n+1} = g^{(n+1)}(0) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(0) = \sum_{k=0}^{n} \binom{n}{k} b_k.$$

This proves (2).

Next we recall:

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

Thus:

$$e^{(e^x)} = \sum_{k=0}^{\infty} \frac{e^{kx}}{k!} = \sum_{k=0}^{\infty} \frac{\sum\limits_{n=0}^{\infty} \frac{(kx)^n}{n!}}{k!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{k^n}{k!} \right) \frac{x^n}{n!}.$$

We note here that interchanging the order of summation in the above is allowed because the series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all values of x. Thus:

$$e^{(e^x - 1)} = e^{(e^x)} e^{-1} = \sum_{n=0}^{\infty} \left(\frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \right) \frac{x^n}{n!}.$$

Comparing this with (1) we see this proves (3). □