SOLUTION FOR MAY 2023

Correct solutions were submitted by:

Ria Garg
Zachary Li - Winner
Victor Lin
Eric Peng - Runner-Up

Solution:

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = 2\pi.$$

Proof: First notice if we make the change of variables $u = t - \pi$ then:

$$\int_\pi^{2\pi} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = \int_0^{\pi} \frac{ab}{a^2\cos^2 u + b^2 \sin^2 u}\,du$$

and thus:

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = 2\int_0^{\pi} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt.$$

Next if we make the change of variables $u = \pi - t$ then:

$$\int_{\frac{\pi}{2}}^{\pi} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = \int_0^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt.$$

Thus:

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = 4\int_0^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt.$$

Next we rewrite the integral and make the substitution $u = \tan(t)$ and $du = \sec^2 t\,dt$ to obtain:

$$4\int_0^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = 4\int_0^{\frac{\pi}{2}} \frac{ab\sec^2 t}{a^2 + b^2\tan^2 t}\,dt$$

$$= 4\int_0^{\infty} \frac{ab}{a^2 + b^2u^2}\,du = 4\tan^{-1}\left(\frac{bu}{a}\right)\Big|_0^{\infty} = 4\frac{\pi}{2} = 2\pi.$$