SOLUTION FOR OCTOBER 2022

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Determine all continuous functions which satisfy:

\[ f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \]

for all real \( x, y \).

**SOLUTION:** \( f(x) = ax + b \) for some constants \( a, b \).

**Proof:** First let \( h(x) = f(x) - f(0) \). Then note that \( h \) is continuous and \( h(0) = 0 \). Also we see:

\[
h\left(\frac{x+y}{2}\right) = f\left(\frac{x+y}{2}\right) - f(0) = \frac{f(x) + f(y) - f(0)}{2} = \frac{h(x) + h(y)}{2}.
\]

Now substituting \( y = 0 \) gives:

\[
h\left(\frac{x}{2}\right) = \frac{h(x)}{2}.
\]

It then follows from this that:

\[
h\left(\frac{x+y}{2}\right) = \frac{h(x+y)}{2}.
\]

Now using (1) we see that:

\[
h(x+y) = h(x) + h(y).
\]

It follows by induction that \( h(nx) = nh(x) \) for all integers \( n \) and replacing \( x \) with \( \frac{x}{n} \) gives \( h(x/n) = (1/n)h(x) \). Now replacing \( x \) with \( mx \) where \( m \) is an integer and using the previous two equations gives \( h(m/n) = \frac{m}{n}h(x) \). Thus:

\[
h(rx) = rh(x) \text{ for all rational } r.
\]

Finally for a general real number \( r \) we can find a sequence of rationals \( r_n \) such that \( \lim_{n \to \infty} r_n = r \). Now using (2) and the fact that \( h \) is continuous then it follows:

\[
h(rx) = \lim_{n \to \infty} h(r_n x) = \lim_{n \to \infty} r_n h(x) = rh(x).
\]
Recalling that $h(r) = f(r) - f(0)$, letting $x = 1$ and $r$ an arbitrary real number gives $h(r) = rh(1) = [f(1) - f(0)]r$ and so we finally see:

$$f(x) = f(0) + [f(1) - f(0)]x.$$