

### SOLUTION FOR JANUARY 2015

Find all continuous nonnegative functions which satisfy:

$$f(x+t) = f(x) + f(t) + 2\sqrt{f(x)}\sqrt{f(t)} \text{ for } x \geq 0, t \geq 0. \quad (1)$$

**SOLUTION:**

$$f(x) = cx^2 \text{ where } c \geq 0.$$

**Proof:** First, notice that we may rewrite (1) as:

$$f(x+t) = \left(\sqrt{f(x)} + \sqrt{f(t)}\right)^2$$

and so since  $f(x)$  is nonnegative:

$$\sqrt{f(x+t)} = \sqrt{f(x)} + \sqrt{f(t)}.$$

Denoting  $g(x) = \sqrt{f(x)}$  we see then that:

$$g(x+t) = g(x) + g(t) \text{ for } x \geq 0, t \geq 0. \quad (2)$$

Substituting  $x = t = 0$  into (2) we see that:

$$g(0) = 2g(0)$$

and so  $g(0) = 0$ . Substituting  $x = t$  into (2) gives:

$$g(2t) = 2g(t).$$

It follows by induction that:

$$g(mt) = mg(t) \quad (3)$$

for every positive integer  $m$ . Letting  $t = \frac{1}{m}$  in (3) gives:

$$g\left(\frac{1}{m}\right) = \frac{1}{m}g(1). \quad (4)$$

Letting  $t = \frac{1}{n}$  in (3) and (4) gives:

$$g\left(\frac{m}{n}\right) = mg\left(\frac{1}{n}\right) = \frac{m}{n}g(1).$$

Thus

$$g(r) = g(1)r$$

for every nonnegative rational number  $r$ . Finally since the rational numbers are dense in the set of real numbers and since we assumed  $f$  is continuous, it follows that  $g$  is continuous and therefore it must be that:

$$g(x) = g(1)x$$

for every nonnegative real number  $x$ . Denoting  $d = g(1)$  we obtain:

$$\sqrt{f(x)} = g(x) = dx$$

for every nonnegative real number  $x$ . Hence:

$$f(x) = cx^2$$

for every nonnegative real number  $x$  where  $c = d^2 \geq 0$ .