PROBLEM OF THE MONTH
OCTOBER 2014 - SOLUTION

Determine:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots. \]

Solution:

\[ \frac{\ln(2)}{3} + \frac{\pi}{3^2}. \]

You may recall the formula for a finite geometric series which says for \( x \neq 1 \):

\[ 1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}. \]

Replacing \( x \) by \( -x^3 \) gives for \( x \neq 1 \):

\[ 1 - x^3 + x^6 - x^9 + \cdots + (-1)^{n+1}x^{3n-3} = \frac{1 - (-1)^n x^{3n}}{1 + x^3}. \]

Integrating on \([0, 1]\) gives:

\[ \int_0^1 \frac{(-1)^n x^{3n}}{1 + x^3} \, dx \]

Next we observe since \( 0 \leq x \leq 1 \) that:

\[ \left| \int_0^1 \frac{(-1)^n x^{3n}}{1 + x^3} \, dx \right| \leq \int_0^1 \frac{x^{3n}}{1 + x^3} \, dx \leq \int_0^1 x^{3n} \, dx = \frac{1}{3n+1} \to 0 \text{ as } n \to \infty. \]

Thus we see by taking limits in (1) that:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} = \int_0^1 \frac{1}{1 + x^3} \, dx. \]

Now using partial fractions:

\[ \int_0^1 \frac{1}{1 + x^3} \, dx = \frac{1}{3} \int_0^1 \left( \frac{1}{1 + x} + \frac{-x + 2}{x^2 - x + 1} \right) \, dx \]

\[ = \frac{1}{3} \left( \ln(x) - \frac{1}{2} \ln(x^2 - x + 1) + \sqrt{3} \tan^{-1}\left( \frac{2}{\sqrt{3}} (x - \frac{1}{2}) \right) \right) \bigg|_0^1 \]

\[ = \frac{\ln(2)}{3} + \frac{2\sqrt{3}}{3} \tan^{-1}\left( \frac{1}{\sqrt{3}} \right) = \frac{\ln(2)}{3} + \frac{\pi}{3}. \]