PROBLEM OF THE MONTH
NOVEMBER 2014 - SOLUTION

Determine:

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x}\, dx \quad \text{when } a > 0.$$

Solution:

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x}\, dx = \frac{\pi}{2}\ln(a) \text{ when } a > 0.$$

First we rewrite:

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x}\, dx = \lim_{b\to\infty}\left(\int_0^b \frac{\tan^{-1}(ax)}{x}\, dx - \int_0^b \frac{\tan^{-1}(x)}{x}\, dx\right). \tag{1}$$

Making the change of variables in the first integral:

$$u = ax, du = a\, dx$$

gives:

$$\int_0^b \frac{\tan^{-1}(ax)}{x}\, dx = \int_0^{ab} \frac{\tan^{-1}(u)}{u}\, du.$$

Therefore we can rewrite (1) as:

$$\lim_{b\to\infty}\left(\int_0^b \frac{\tan^{-1}(ax)}{x}\, dx - \int_0^b \frac{\tan^{-1}(x)}{x}\, dx\right) = \lim_{b\to\infty}\int_b^{ab} \frac{\tan^{-1}(x)}{x}\, dx.$$

Now we make another change of variables:

$$x = by, dx = b\, dy$$

which gives:

$$\lim_{b\to\infty}\int_b^{ab} \frac{\tan^{-1}(x)}{x}\, dx = \lim_{b\to\infty}\int_1^{a} \frac{\tan^{-1}(by)}{y}\, dy. \tag{2}$$

Finally, since $\tan^{-1}(by) \to \frac{\pi}{2}$ uniformly on $[1,\infty) \cup [a,\infty)$ then from (1)-(2):

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x}\, dx = \lim_{b\to\infty}\int_1^{a} \frac{\tan^{-1}(by)}{y}\, dy = \frac{\pi}{2}\int_1^a \frac{1}{y}\, dy = \frac{\pi}{2}\ln(a).$$