SOLUTION FOR NOVEMBER 2016

Prove that the polynomial \( p(x) = x^3 - 12x^2 + ax - 64 \) has all of its roots real and nonnegative for exactly one real number \( a \). Determine \( a \).

SOLUTION:

\[ a = 48 \] and thus \( p(x) = (x - 4)^3 \).

Let \( p(x) = x^3 - 12x^2 + ax - 64 \) and let us assume that \( p(x) \) has all of its roots real and nonnegative. So we can write:

\[ p(x) = (x - r_1)(x - r_2)(x - r_3) \quad \text{with} \quad 0 \leq r_1 \leq r_2 \leq r_3. \]

Multiplying this out and equating coefficients gives:

\[ r_1 + r_2 + r_3 = 12, \quad r_1r_2 + r_1r_3 + r_2r_3 = a, \quad r_1r_2r_3 = 64. \]

From the equation \( r_1r_2r_3 = 64 \) we see that in fact that \( r_1, r_2, r_3 \) must be strictly positive and then using this in the equation \( r_1r_2 + r_1r_3 + r_2r_3 = a \) implies that \( a > 0 \).

Next we see that:

\[ p'(x) = 3x^2 - 24x + a. \]

If \( p'(x) \geq 0 \) for all \( x \) then \( p \) has only one real root and so it must be that \( r_1 = r_2 = r_3 \) and therefore \( 3r_1 = 12 \) and so \( r_1 = r_2 = r_3 = 4 \) and thus \( 48 = 16 + 16 + 16 = a \) and so we are done in this case. So now suppose \( p'(x) \) gets negative. Then since \( p' \) is a quadratic then \( p'(x) = 0 \) has two real solutions.

From the quadratic formula we see that \( p'(x) = 0 \) at:

\[ 4 \pm \sqrt{48 - a}. \]

Thus we must have \( 48 - a \geq 0 \), i.e. \( a \leq 48 \). Thus we see \( 0 < a \leq 48 \). Further \( p(x) \) has a local maximum at \( 4 - \sqrt{48-a} \) and a local minimum at \( 4 + \sqrt{48-a} \).

Next we rewrite \( p(x) \) as:

\[ p(x) = (x-4)^3 + (a-48)x. \]

Then we calculate and see that if \( 0 < a < 48 \) then:

\[ p\left(4 - \frac{\sqrt{48-a}}{3}\right) = \frac{2(48-a)}{3} \left[ \sqrt{\frac{48-a}{3}} - 6 \right] < 0 \quad \text{since} \quad 0 < a < 48. \]

Then it follows that \( p(x) \) has only one real root and as earlier this implies \( r_1 = r_2 = r_3 = 4 \) and thus \( 48 = 16 + 16 + 16 = a \). This contradicts that \( 0 < a < 48 \) and therefore our assumption that \( 0 < a < 48 \) must be false. From earlier we know \( a > 0 \) so it must be that \( a \geq 48 \). From earlier we also know \( a \leq 48 \) therefore it must be that \( a = 48 \). This completes the proof.