SOLUTION FOR DECEMBER 2015

Determine if the following series converges:

$$1 + \frac{1}{2}\left(\frac{19}{7}\right) + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \cdots$$

SOLUTION: The series does indeed converge.

Notice that if we denote $x = \frac{19}{7}$ then we can rewrite the series as:

$$1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^n}x^n.$$

Applying the ratio test we see that:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)!|x|^{n+1}}{(n+2)^{n+1}}}{\frac{n!|x|^n}{(n+1)^n}} = \frac{(n+1)(n+1)^n|x|}{(n+2)^{n+1}}$$

$$= \frac{(n+1)^{n+1}|x|}{(n+2)^{n+1}} = \left(1 - \frac{1}{n+2}\right)^{n+1}|x| = \frac{\left(1 - \frac{1}{n+2}\right)^{n+2}}{\left(1 - \frac{1}{n+2}\right)}|x|.$$

You might recall from calculus that:

$$\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$$

and in particular that:

$$\lim_{n\to\infty}(1 - \frac{1}{n})^n = e^{-1}.$$

And so it follows that:

$$\lim_{n\to\infty} \frac{\left(1 - \frac{1}{n+2}\right)^{n+2}}{\left(1 - \frac{1}{n+2}\right)}|x| = e^{-1}|x|.$$

Thus the above series converges when:

$$e^{-1}|x| < 1, \text{ i.e. when } |x| < e$$

and since $x = \frac{19}{7} < e$ then we see that the given series converges by the ratio test. (Note that $\frac{19}{7} = 2.714...$ and $e = 2.718...$).