SOLUTION FOR DECEMBER 2015

Determine if the following series converges:

\[ 1 + \frac{1}{2} \left( \frac{19}{7} \right) + \frac{2!}{3^2} \left( \frac{19}{7} \right)^2 + \frac{3!}{4^3} \left( \frac{19}{7} \right)^3 + \cdots \]

**SOLUTION:** The series does indeed converge.

Notice that if we denote \( x = \frac{19}{7} \) then we can rewrite the series as:

\[ 1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^n} x^n. \]

Applying the ratio test we see that:

\[
\frac{|a_{n+1}|}{a_n} = \frac{(n+1)|x|^{n+1}}{(n+2)^{n+1}} \frac{(n+1)(n+1)^n|x|}{n!(n+1)^n} = \frac{(n+1)(n+1)^n|x|}{(n+2)^{n+1}}
\]

\[ = \frac{(n+1)^{n+1}|x|}{(n+2)^{n+1}} = \left(1 - \frac{1}{n+2}\right)^{n+1} |x| = \frac{(1 - \frac{1}{n+2})^{n+2} - \frac{1}{n+2}}{(1 - \frac{1}{n+2})} |x|. \]

You might recall from calculus that:

\[ \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \]

and in particular that:

\[ \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}. \]

And so it follows that:

\[ \lim_{n \to \infty} \left(1 - \frac{1}{n+2}\right)^{n+2} |x| = e^{-1}|x|. \]

Thus the above series converges when:

\[ e^{-1}|x| < 1, \text{ i.e. when } |x| < e \]

and since \( x = \frac{19}{7} < e \) then we see that the given series converges by the ratio test. (Note that \( \frac{19}{7} = 2.714... \text{ and } e = 2.718...\)).