SOLUTION FOR OCTOBER 2017

Determine:

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}.$$

Hint: You may assume there exists a constant A such that:

$$\lim_{n\to\infty} (\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \cdots + \frac{\ln(n)}{n} - \frac{1}{2}\ln^2(n)) = A.$$

SOLUTION:

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n} = \gamma \ln(2) - \frac{1}{2}\ln^2(2)$$

where γ is Euler's constant i.e. $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right)$.

It follows from the Alternating Series Test that $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges so let us denote:

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n} = B. \tag{1}$$

Using the hint we see that:

$$\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \cdots + \frac{\ln(2n)}{2n} - \frac{1}{2}\ln^2(2n) = A + d_n \tag{2}$$

where $d_n \to 0$ as $n \to \infty$. From (1) we also have:

$$\frac{\ln(2)}{2} - \frac{\ln(3)}{3} + \cdots - \frac{\ln(2n-1)}{2n-1} + \frac{\ln(2n)}{2n} = B + c_n \tag{3}$$

where $c_n \to 0$ as $n \to \infty$ Adding (2)-(3) gives:

$$\ln(2) + \frac{\ln(4)}{2} + \cdots + \frac{\ln(2n)}{n} - \frac{1}{2}\ln^2(2n) = A + B + h_n \tag{4}$$

where $h_n = d_n + c_n$. Rewriting (4) we obtain:

$$\ln(2) + \frac{\ln(2)+\ln(2)}{2} + \cdots + \frac{\ln(2)+\ln(n)}{n} - \frac{1}{2}\ln^2(2n) = A + B + h_n$$

and this is:

$$\ln(2)\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right) + \left(\frac{\ln(2)}{2} + \cdots + \frac{\ln(n)}{n}\right) - \frac{1}{2}\ln^2(2n) = A + B + h_n. \tag{5}$$

Now writing:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} = \ln(n) + \gamma + k_n \tag{6}$$

where γ is Euler's constant and $k_n \to 0$ as $n \to \infty$ and also writing:

$$\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \cdots + \frac{\ln(n)}{n} = \frac{1}{2}\ln^2(n) + A + j_n \tag{7}$$

where $j_n \to 0$ as $n \to \infty$ and substituting into (5) gives:

$$\ln(2)(\ln(n) + \gamma + k_n) + \frac{1}{2}\ln^2(n) + A + j_n - \frac{1}{2}\ln^2(2n) = A + B + l_n$$

where $l_n \to 0$ as $n \to \infty$.

Subtracting A from both sides gives:

$$\ln(2)(\ln(n) + \gamma) + \frac{1}{2}\ln^2(n) - \frac{1}{2}\ln^2(2n) = B + p_n \tag{8}$$

where $p_n = l_n - j_n - \ln(2)k_n$. Since l_n, j_n and $k_n \to 0$ as $n \to \infty$ we see that $p_n \to 0$ as $n \to \infty$.

Next by rules of logs:

$$\frac{1}{2}\ln^2(2n) = \frac{1}{2}(\ln(2) + \ln(n))^2 = \frac{1}{2}\ln^2(2) + \ln(2)\ln(n) + \frac{1}{2}\ln^2(n)$$

and therefore substituting this into (8) gives:

$$B = \gamma\ln(2) - \frac{1}{2}\ln^2(2) + p_n.$$

Finally since p_n can be made arbitrarily small for sufficiently large n we see that:

$$B = \gamma\ln(2) - \frac{1}{2}\ln^2(2).$$