# SOLUTION FOR NOVEMBER 2017

Let:

$$x_{n+1} = \frac{1}{n+1} x_{n-1} + \frac{n}{n+1} x_n$$

with:

$$x_0 = a, x_1 = b.$$

Determine if:

$$\lim_{n\to\infty} x_n \text{exists}$$

and if so then find the limit.

**SOLUTION:**

$$\lim_{n\to\infty} x_n = ae^{-1} + b(1 - e^{-1}).$$

We first notice that:

$$x_{n+1} - x_n = \frac{-1}{n+1}(x_n - x_{n-1}).$$

One can then show by induction that:

$$x_n - x_{n-1} = \frac{(-1)^{n-1}}{n!}(x_1 - x_0).$$

Now summing from 1 to $N$ gives:

$$x_N - x_0 = (x_1 - x_0) \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n!}.$$

Thus:

$$x_N = a + (b - a) \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n!} = a - (b - a) \sum_{n=1}^{N} \frac{(-1)^n}{n!}.$$

We now recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and therefore:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \text{ and so } e^{-1} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

Therefore taking limits in the $x_N$ equation gives:

$$\lim_{N\to\infty} x_N = a - (b - a)(e^{-1} - 1) = ae^{-1} + b(1 - e^{-1}).$$