SOLUTION FOR SEPTEMBER 2017

Denoting the diameters as d_n, determine d_n and $\sum_{n=1}^{\infty} d_n$.

SOLUTION: $d_n = \frac{1}{n(n+1)}$ and $\sum_{n=1}^{\infty} d_n = 1$.

Let A be the center of the left-most circle with radius 1. Let B be the point of tangency between the two circles of radius 1. Let D_n be the center of the circle with diameter d_n. Then the triangle ABD_n is a right triangle with sides of length 1, $1-(d_1+d_2+\cdots+d_{n-1})-\frac{1}{2}d_n$, and $1+\frac{1}{2}d_n$. Then by the Pythagorean theorem:

$$1^2 + \left(1-(d_1+d_2+\cdots+d_{n-1})-\frac{1}{2}d_n\right)^2 = (1+\frac{1}{2}d_n)^2.$$

Thus:

$$1+(1-(d_1+d_2+\cdots+d_{n-1}))^2 - d_n(1-(d_1+d_2+\cdots+d_{n-1}) + \frac{1}{4}d_n^2 = 1+d_n+\frac{1}{4}d_n^2.$$

Therefore:

$$(1-(d_1+d_2+\cdots+d_{n-1}))^2 - d_n(1-(d_1+d_2+\cdots+d_{n-1}) = d_n.$$

Rearranging and solving for d_n gives:

$$d_n = \frac{(1-(d_1+d_2+\cdots+d_{n-1}))^2}{2-(d_1+d_2+\cdots+d_{n-1})}.$$

Notice then that $d_1 = 1/2$.

It can then be shown by induction that $d_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Next notice that we obtain a telescoping sum $\sum_{n=1}^{N} d_n = \sum_{n=1}^{N} [\frac{1}{n} - \frac{1}{n+1}] = 1 - \frac{1}{N+1}$.

Finally we see $\sum_{n=1}^{\infty} d_n = \lim_{N\to\infty} [1 - \frac{1}{N+1}] = 1$.