SOLUTION FOR OCTOBER 2018

Let n be a nonzero integer. Multiply this by 3 (write your answer in base 10). Now add the digits together and call this number m (written in base 10). If $0 \leq m \leq 9$ then this process stops. Otherwise add the digits of m together to get a new number p (written in base 10). As above if $0 \leq p \leq 9$ then stop but otherwise continue this process until obtaining just one digit. Make a conjecture about what this last number is and then prove it.

SOLUTION: The number is either 3, 6, or 9.

Proof: Let n be a nonzero integer. Let us write $3n = a_m 10^m + a_{m-1}10^{m-1} \cdots + a_1 10 + a_0$ where the a_i are all integers with $0 \leq a_i \leq 9$. Rewriting we obtain:

$$3n = a_m\left((10^m - 1) + 1\right) + a_{m-1}((10^{m-1} - 1) + 1) + \cdots + a_1((10 - 1) + 1) + a_0$$

$$= \big(a_m(10^m - 1) + a_{m-1}(10^{m-1} - 1) + \cdots + a_1(10 - 1)\big) + (a_m + a_{m-1} + \cdots a_1 + a_0)\,. \tag{1}$$

Now each of the terms $(10-1) = 9, (10^2-1) = 99, (10^3-1) = 999, \cdots$ is divisible by 3 (one can prove this by induction) and since the left-hand side of (1) is also divisible by 3 it follows from (1) that $(a_m + a_{m-1} + \cdots a_1 + a_0)$ is divisible by 3 and nonzero since n is nonzero. If $0 < (a_m + a_{m-1} + \cdots a_1 + a_0) < 10$ then we see that we get a divisible by 3 and so this number must be 3, 6, or 9. If $(a_m + a_{m-1} + \cdots a_1 + a_0) \geq 10$ then we apply this same procedure but now this time starting with $3n = a_m + a_{m-1} + \cdots a_1 + a_0$. Again we see after each application of this procedure that we will get a number divisible by 3. So eventually we will get an integer between 0 and 10 that is divisible by 3 so then the final number must be 3, 6, or 9.

Note: All three of the numbers 3, 6, and 9 do actually come up when applying this process. For example, if we start with $n = 8$ then $3n = 24$ and then adding the digits of 24 we get 2+4 = 6. If we start with $n = 22$ then $3n = 66$ and adding the digits gives 6+6 =12. Adding digits again gives 1+2=3. Finally if we start with $n = 15$ we get $3n = 45$ and adding digits gives 9.