SOLUTION FOR NOVEMBER 2018

Let $0 < a < b$. Then:

$$\int_a^b \cos^{-1}\left(\frac{x}{\sqrt{(a+b)x-ab}}\right) = \frac{(b-a)^2\pi}{4(a+b)}.$$

Proof: We first integrate by parts and after simplifying we get:

$$x\cos^{-1}\left(\frac{x}{\sqrt{(a+b)x-ab}}\right)\Bigg|_a^b + \int_a^b \frac{x[(\frac{a+b}{2})x-ab]}{\sqrt{(a+b)x-ab-x^2}[(a+b)x-ab]}\,dx.$$

The first two terms evaluate to 0 and so we just have to calculate the integral in the above.

Completing the square under the square root sign gives:

$$\int_a^b \frac{x[(\frac{a+b}{2})x-ab]}{\sqrt{(\frac{b-a}{2})^2-(x-\frac{a+b}{2})^2}[(a+b)x-ab]}\,dx.$$

Now letting $u = x - \frac{a+b}{2}$ gives:

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{(u+\frac{a+b}{2})\left(\frac{a+b}{2}(u+\frac{a+b}{2})-ab\right)}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)(u+\frac{a+b}{2})-ab]}\,du.$$

Rewriting we obtain:

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{(u+\frac{a+b}{2})\left(\frac{a+b}{2}(u+\frac{a+b}{2})-ab\right)}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)(u+\frac{a+b}{2})-ab]}\,du = \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{(u+\frac{a+b}{2})\left(\frac{a+b}{2}(u+\frac{a+b}{2})-\frac{ab}{2}-\frac{ab}{2}\right)}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)(u+\frac{a+b}{2})-ab]}\,du$$

$$= \frac{1}{2}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{u+\frac{a+b}{2}}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du - \frac{ab}{2}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{(u+\frac{a+b}{2})}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)(u+\frac{a+b}{2})-ab]}\,du$$

$$= \frac{1}{2}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{u+\frac{a+b}{2}}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du - \frac{ab}{2(a+b)}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{(a+b)(u+\frac{a+b}{2})-ab+ab}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)(u+\frac{a+b}{2})-ab]}\,du$$

$$= \frac{1}{2}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{u+\frac{a+b}{2}}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du - \frac{ab}{2(a+b)}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du - \frac{(ab)^2}{2(a+b)}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)(u+\frac{a+b}{2})-ab]}\,du$$

$$= \frac{1}{2}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{u}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du + \left(\frac{a+b}{4} - \frac{ab}{2(a+b)}\right)\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du$$

$$- \frac{(ab)^2}{2(a+b)}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)u+\frac{a^2+b^2}{2})]}\,du.$$

The first integral is 0 since it is the integral of an odd function over a symmetric interval around the origin. Thus our integral becomes:

$$\frac{a^2+b^2}{4(a+b)}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du - \frac{(ab)^2}{2(a+b)}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}[(a+b)u+\frac{a^2+b^2}{2})]}\,du. \tag{1}$$

Next:

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{(\frac{b-a}{2})^2-u^2}}\,du = \sin^{-1}(1)-\sin^{-1}(-1)=\pi. \tag{2}$$

For the second integral we let $u=\frac{b-a}{2}v$. This integral becomes:

$$\int_{-1}^{1} \frac{2}{\sqrt{1-v^2}\,((b^2-a^2)v+(b^2+a^2))}\,dv.$$

Now let $v=\cos(\theta)$ and this integral becomes:

$$\frac{2}{b^2+a^2}\int_0^{\pi} \frac{1}{1+\frac{b^2-a^2}{b^2+a^2}\cos(\theta)}\,d\theta. \tag{3}$$

Now let $c=\frac{b^2-a^2}{b^2+a^2}$ and note that $0<c<1$. Finally we will show that:

$$\int_0^{\pi} \frac{1}{1+c\cos(\theta)}\,d\theta = \frac{\pi}{\sqrt{1-c^2}}. \tag{4}$$

Then using (2)-(4) and substituting into (1) we obtain:

$$\frac{(a^2+b^2)\pi}{4(a+b)} - \frac{a^2b^2}{2(a+b)}\frac{2}{(b^2+a^2)}\frac{\pi}{\sqrt{1-\frac{(b^2-a^2)^2}{(b^2+a^2)^2}}} = \frac{(a^2+b^2)\pi}{4(a+b)} - \frac{a^2b^2}{a+b}\frac{\pi}{\sqrt{4a^2b^2}} = \frac{\pi}{4(a+b)}(a^2+b^2-2ab) =$$

$$\frac{(b-a)^2\pi}{4(a+b)}. \quad \square$$

To show (4) we make the substitution:

$$u=\tan(\theta/2)=\frac{\sin(\theta)}{1+\cos(\theta)}.$$

Then:

$$\frac{2}{1+u^2}\,du = d\theta; \quad \cos(\theta)=\frac{1-u^2}{1+u^2}$$

and so the left side of (4) becomes:

$$\frac{2}{1+c}\int_0^{\infty} \frac{1}{1+\frac{1-c}{1+c}u^2}\,du = \frac{2}{\sqrt{1-c^2}}\tan^{-1}\left(\sqrt{\frac{1-c}{1+c}}u\right)\Big|_0^{\infty} = \frac{2}{\sqrt{1-c^2}}\frac{\pi}{2} = \frac{\pi}{\sqrt{1-c^2}}.$$