SOLUTION FOR FEBRUARY 2018

Determine the values of $\theta \in [0, \pi]$ for which:

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\theta)}{n} \quad \text{converges.}$$

SOLUTION: The series only converges at $\theta = 0$ and $\theta = \pi$.

First of all if $\theta = 0$ and $\theta = \pi$ then each term in the series is 0 and so the series converges to 0.

So now let us suppose $0 < \theta < \pi$. Let:

$$a_n = \sin^2(n\theta); \quad b_n = \frac{1}{n}.$$

Also let:

$$A_n = \sum_{k=1}^{n} a_n; \quad S_n = \sum_{k=1}^{n} a_n b_n.$$

Then it is straightforward to show for $q > p$:

$$S_q - S_p = \sum_{k=p}^{q} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p \tag{1}$$

and:

$$A_n = \sum_{k=1}^{n} a_n = \sum_{k=1}^{n} \sin^2(n\theta) = \frac{1}{2} \sum_{k=1}^{n} (1 + \cos(2n\theta)).$$

It is straightforward to show:

$$\sum_{k=1}^{n} \cos(2k\theta) = \frac{\sin((2n+1)\theta)}{2\sin(\theta)} + \frac{1}{2}.$$

Therefore:

$$A_n = \frac{n+1}{2} + \frac{\sin((2n+1)\theta)}{2\sin(\theta)}.$$

Thus since $\sin((2n+1)\theta) \geq -1$ and since $\sin(\theta) > 0$ for $0 < \theta < \pi$ we get:

$$A_n \geq \frac{n+1}{2} - \frac{1}{2\sin(\theta)}$$

and so for any fixed value of θ with $0 < \theta < \pi$ then $\sin(\theta) > 0$ and therefore for $n \geq n_0$ where n_0 is sufficiently large:

$$A_n \geq \frac{n}{4}.$$

Then returning to (1) and assuming $q \geq n_0$ we obtain:

$$S_q - S_{n_0} \geq \sum_{k=n_0}^{q} \frac{k}{4}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{4} - A_{n_0-1}b_{n_0} = \sum_{k=n_0}^{q} \frac{1}{4(k+1)} + \frac{1}{4} - A_{n_0-1}b_{n_0} \to \infty \text{ as } q \to \infty.$$

Thus $S_q \to \infty$ as $q \to \infty$. Therefore:

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\theta)}{n} \text{ diverges for } 0 < x < \pi.$$