SOLUTION FOR MARCH 2018

Let $x > 1$ and define $f(e) = e$. For $x > 1$ and $x \neq e$ let $f(x) > 0$ be the unique number such that $f(x) \neq x$ and $x^{f(x)} = f(x)^x$. Sketch the graph of $f(x)$ and calculate $f'(x)$ (you may leave this answer in terms of x and $f(x)$). Finally calculate $f'(e)$.

SOLUTION: Rewriting: $x^{f(x)} = f(x)^x$ we see that:

$$f(x)^{\frac{1}{f(x)}} = x^{\frac{1}{x}}. \tag{1}$$

So let us examine the function: $g(x) = x^{1/x}$. Taking natural log we get: $\ln(g) = \frac{\ln(x)}{x}$ and then differentiating gives: $\frac{g'}{g} = \frac{1-\ln(x)}{x^2}$ and so: $g'(x) = x^{1/x}\left(\frac{1-\ln(x)}{x^2}\right)$. So we see that $g(x)$ has exactly one criticial point at $x = e$, this is a local maximum, and $g(e) = e^{1/e}$. Also $\lim\limits_{x\to 0^+} g(x) = 0$ and $\lim\limits_{x\to\infty} g(x) = 1$.

Now let $1 < y_0 < e^{1/e}$. Then we see that there are exactly two solutions x_1, x_2 with $1 < x_1 < e < x_2 < \infty$ of $g(x) = y_0$.

Thus it follows that: $x_2 = f(x_1)$ (and also $x_1 = f(x_2)$ and thus $f(f(x)) = x$).

We see that for $1 < x < \infty$ as x increases then $f(x)$ decreases. In addition:

$$\lim_{x\to 1^+} f(x) = \infty; \text{ and } \lim_{x\to\infty} f(x) = 1.$$

Taking natural logs and differentiating (1) gives:

$$\frac{f'(x) - f'(x)\ln f(x)}{f^2(x)} = \frac{1-\ln(x)}{x^2}.$$

Therefore:

$$f'(x) = \frac{1-\ln(x)}{1-\ln f(x)}\frac{f^2(x)}{x^2}. \tag{2}$$

This defines $f'(x)$ for $1 < x < \infty$ and $x \neq e$. Differentiating $f(f(x)) = x$ gives: $f'(f(x))f'(x) = 1$ and so evaluating at $x = e$ we obtain $f'^2(e) = 1$ and since f is decreasing we see that:

$$f'(e) = -1.$$

(One could also take limits in (2) and use L'Hopital's rule).