SOLUTION FOR MAY 2018

Let \( a > 0 \) and \( b > 0 \). Determine:

\[
\int_0^{2\pi} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt.
\]

**SOLUTION:**

\[
\int_0^{2\pi} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt = 2\pi.
\]

Since the integrand is \( 2\pi \) periodic we can integrate over any interval of length \( 2\pi \) so:

\[
\int_0^{2\pi} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt.
\]

Next since \( \cos^2(t + \pi) = \cos^2(t) \) and \( \sin^2(t + \pi) = \sin^2(t) \) it follows that:

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt.
\]

Also since \( \cos^2(t) \) and \( \sin^2(t) \) are even functions it follows that:

\[
2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt = 4 \int_0^{\frac{\pi}{2}} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt.
\]

Next dividing top and bottom by \( \cos^2(t) \) we obtain:

\[
4 \int_0^{\frac{\pi}{2}} \frac{ab}{a^2 \cos^2(t) + b^2 \sin^2(t)} \, dt = 4 \int_0^{\frac{\pi}{2}} \frac{ab \sec^2(t)}{a^2 + b^2 \tan^2(t)} \, dt.
\]

Now let \( u = \tan(t) \) so \( du = \sec^2(t) \, dt \) thus:

\[
4 \int_0^{\frac{\pi}{2}} \frac{ab \sec^2(t)}{a^2 + b^2 \tan^2(t)} \, dt = 4 \int_0^\infty \frac{ab}{a^2 + b^2 u^2} \, du
\]

\[
= \lim_{u \to \infty} 4 \tan^{-1} \left( \frac{b}{a} \right) = 4 \frac{\pi}{2} = 2\pi.
\]