SOLUTION FOR SEPTEMBER 2018

Investigate the convergence of:

$$\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)]^{\ln n}}$$

and:

$$\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)]^{\ln(\ln n)}}.$$

SOLUTION:

$$\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)]^{\ln n}} = \text{ converges}$$

and:

$$\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)]^{\ln(\ln n)}} = \text{ diverges.}$$

The winner for September is Rhythm Garg. He showed that:

$$\frac{1}{[\ln(\ln n)]^{\ln n}} \leq \frac{1}{n^2} \quad \text{for large} \quad n$$

and since $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges (it is a p-series with $p = 2$) therefore:

$$\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)]^{\ln n}} \text{ converges by comparison with } \sum_{n=3}^{\infty} \frac{1}{n^2}.$$

Similarly he showed

$$\frac{1}{[\ln(\ln n)]^{\ln(\ln n)}} \geq \frac{1}{n} \quad \text{for large} \quad n$$

and since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (it is a p-series with $p = 1$) therefore:

$$\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)]^{\ln n}} \text{ diverges by comparison with } \sum_{n=3}^{\infty} \frac{1}{n}.$$

Another way to do this is to use the Integral Test. The first would be:

$$\int_3^\infty \frac{1}{[\ln(\ln x)]^{\ln x}}\, dx.$$

Let $u = \ln(x)$ and thus $e^u\, du = dx$ then we get:

$$\int_{\ln(3)}^\infty \left(\frac{e}{\ln u}\right)^u du.$$

Now $\lim_{u\to\infty} \frac{e}{\ln u} = 0$ so for large u say $u \geq u_0$ we have: $\frac{e}{\ln(u)} \leq \frac{1}{e} = e^{-1}$ therefore:

$$\int_{u_0}^\infty \left(\frac{e}{\ln u}\right)^u du \leq \int_{u_0}^\infty e^{-u}\, du = e^{-u_0}.$$

Thus:

$$\sum_{n=3}^\infty \frac{1}{[\ln(\ln n)]^{\ln n}} = \text{ converges.}$$

The other integral to investigate is:

$$\int_3^\infty \frac{1}{[\ln(\ln x)]^{\ln(\ln x)}}\, dx.$$

Similarly making the substituion $u = \ln(x)$ and $e^u\, du = dx$ gives:

$$\int_{\ln(3)}^\infty \left(\frac{e}{(\ln u)^{\frac{\ln u}{u}}}\right)^u du.$$

Now notice that:

$$\lim_{u\to\infty} (\ln u)^{\frac{\ln(u)}{u}} = 1.$$

(This can be shown by taking ln of this expression and using L'Hopital's rule).
Thus:

$$\lim_{u\to\infty} \frac{e}{(\ln u)^{\frac{\ln u}{u}}} = e$$

therefore for large u say $u \geq u_0$ we have:

$$\frac{e}{(\ln u)^{\frac{\ln u}{u}}} \geq 2.$$

Hence:

$$\int_{u_0}^\infty \left(\frac{e}{(\ln u)^{\frac{\ln u}{u}}}\right)^u \geq \int_{u_0}^\infty 2^u\, du = \infty.$$

Thus:

$$\sum_{n=3}^\infty \frac{1}{[\ln(\ln n)]^{\ln(\ln n)}} = \text{ diverges.}$$