SOLUTION FOR OCTOBER 2019

Correct solutions were submitted by:

Weilun Sun
Hubert Yang
Rhythm Garg
David Duhon

SOLUTION: For $0 < x < 1$:

$$\ln(x)\ln(1-x) + \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2} = \frac{\pi^2}{6}. \tag{1}$$

Proof: For $0 < x < 1$ let:

$$f(x) = \ln(x)\ln(1-x) + \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2}. \tag{2}$$

Differentiating the sums term-by-term which is allowed since both sums converge uniformly and the differentiated sums converge uniformly in a neighborhood of x gives:

$$f'(x) = \frac{\ln(1-x)}{x} - \frac{\ln(x)}{1-x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} - \sum_{n=1}^{\infty} \frac{(1-x)^{n-1}}{n}. \tag{3}$$

Now recall for $0 \leq x < 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Integrating on $(0, x)$ gives:

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

Thus:

$$\frac{-\ln(1-x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}. \tag{4}$$

Therefore:

$$\frac{\ln(1-x)}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = 0. \tag{5}$$

Replacing x with $1-x$ gives:

$$\frac{\ln(x)}{1-x} + \sum_{n=1}^{\infty} \frac{(1-x)^{n-1}}{n} = 0. \tag{6}$$

Combining (3), (5)-(6) we see:

$$f'(x) = 0 \text{ (!!!)}$$

and therefore $f(x)$ is constant. Using L'Hopital's rule one can show $\ln(x)\ln(1-x) \to 0$ as $x \to 1^-$ and thus using (2) we have $\lim_{x\to 1^-} f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and so (1) follows.

Note that if we let $x = \frac{1}{2}$ in (1) we obtain:

$$\frac{\pi^2}{6} = f\left(\frac{1}{2}\right) = \ln^2\left(\frac{1}{2}\right) + 2\sum_{n=1}^{\infty} \frac{1}{2^n n^2}$$

hence we obtain the interesting fact:

$$\frac{\pi^2}{12} = \frac{\ln^2(2)}{2} + \sum_{n=1}^{\infty} \frac{1}{2^n n^2}.$$

$\square$