Determine $a_n$ if:

$$a_{n+1} = 2a_n + n \text{ and } a_0 = 1.$$  \hspace{1cm} (1)

**SOLUTION:**

$$a_n = 2^{n+1} - (n + 1).$$

**Proof:** We first use induction to show:

$$a_{n+1} = 2^{n+1} + 2^n \sum_{k=0}^{n} \frac{k}{2^k} \text{ for } n \geq 0. \hspace{1cm} (2)$$

For $n = 0$ we see using (1) that $a_1 = 2a_0 + 0 = 2$ so (2) holds when $n = 0$.

Now assume:

$$a_{n+1} = 2^{n+1} + 2^n \sum_{k=0}^{n} \frac{k}{2^k}. \hspace{1cm} (3)$$

Then using (1) and (3):

$$a_{n+2} = 2a_{n+1} + (n + 1) = 2^{n+2} + 2^{n+1} \sum_{k=0}^{n} \frac{k}{2^k} + (n + 1) = 2^{n+2} + 2^{n+1} \sum_{k=0}^{n+1} \frac{k}{2^k}.$$  

Thus (2) holds by induction.

We will now work on determining:

$$\sum_{k=0}^{n} \frac{k}{2^k}.$$  

For $x \neq 1$ it is known that:

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$
Thus differentiating gives:

\[ \sum_{k=0}^{n} kx^{k-1} = \frac{n x^{n+1} - (n + 1) x^n + 1}{(1 - x)^2} \]

and multiplying by \( x \) gives:

\[ \sum_{k=0}^{n} kx^{k} = \frac{n x^{n+2} - (n + 1) x^{n+1} + x}{(1 - x)^2} \]

Now we let \( x = 1/2 \) and get:

\[ \sum_{k=0}^{n} k \frac{1}{2^k} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}} + \frac{1}{4} = n - \frac{n+1}{2^{n-1}} + 2. \]

Therefore:

\[ 2^n \sum_{k=0}^{n} k \frac{1}{2^k} = n - 2(n + 1) + 2^{n+1} = -n - 2 + 2^{n+1} \]

and finally from (3) we see:

\[ a_{n+1} = 2^{n+2} - (n + 2) \]

and thus:

\[ a_n = 2^{n+1} - (n + 1). \]