SOLUTION FOR NOVEMBER 2019

Correct solutions were submitted by:

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There were 8 other "nearly correct" solutions. To all contestants: please provide a *mathematical proof* of statements that you claim to be true.

Determine a_n if:

$$a_{n+1} = 2a_n + n \text{ and } a_0 = 1. \tag{1}$$

SOLUTION:

$$a_n = 2^{n+1} - (n+1).$$

Proof: We first use induction to show:

$$a_{n+1} = 2^{n+1} + 2^n \sum_{k=0}^{n} \frac{k}{2^k} \text{ for } n \geq 0. \tag{2}$$

For $n = 0$ we see using (1) that $a_1 = 2a_0 + 0 = 2$ so (2) holds when $n = 0$.

Now assume:

$$a_{n+1} = 2^{n+1} + 2^n \sum_{k=0}^{n} \frac{k}{2^k}. \tag{3}$$

Then using (1) and (3):

$$a_{n+2} = 2a_{n+1} + (n+1) = 2^{n+2} + 2^{n+1} \sum_{k=0}^{n} \frac{k}{2^k} + (n+1) = 2^{n+2} + 2^{n+1} \sum_{k=0}^{n+1} \frac{k}{2^k}.$$

Thus (2) holds by induction.

We will now work on determining:

$$\sum_{k=0}^{n} \frac{k}{2^k}.$$

For $x \neq 1$ it is known that:

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Thus differentiating gives:

$$\sum_{k=0}^{n} kx^{k-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}$$

and mutliplying by x gives:

$$\sum_{k=0}^{n} kx^{k} = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.$$

Now we let $x = 1/2$ and get:

$$\sum_{k=0}^{n} \frac{k}{2^k} = \frac{\frac{n}{2^{n+2}} - \frac{n+1}{2^{n+1}} + \frac{1}{2}}{\frac{1}{4}} = \frac{n}{2^n} - \frac{n+1}{2^{n-1}} + 2.$$

Therefore:

$$2^n \sum_{k=0}^{n} \frac{k}{2^k} = n - 2(n+1) + 2^{n+1} = -n - 2 + 2^{n+1}$$

and finally from (3) we see:

$$a_{n+1} = 2^{n+2} - (n+2)$$

and thus:

$$a_n = 2^{n+1} - (n+1).$$

□