SOLUTION FOR MARCH 2019

Solution: The only solution of $a^b = b^a$ with $a < b$ and a, b positive integers is $a = 2$, $b = 4$.

Proof: Taking natural logs of both sides of the equation $a^b = b^a$ and simplifying gives:

$$\frac{\ln(a)}{a} = \frac{\ln(b)}{b}.$$

So we now examine the function $f(x) = \frac{\ln(x)}{x}$ for $x > 0$. It follows then that:

$$f'(x) = \frac{1 - \ln(x)}{x^2}.$$

Thus the only critical point of f occurs when $\ln(x) = 1$. That is, $x = e$. Also notice $\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = 0$ so it follows that $f(x)$ has a maximum at $x = e$, $f(x)$ is increasing on $(0, e)$, and f is decreasing on (e, ∞). In addition, $f(x) < 0$ for $0 < x < 1$.

So we now want to see if there are a, b positive integers such that $f(a) = f(b)$.

Since $f(x)$ has exactly one maximum at $x = e$, then if there are an a and b with $0 < a < b$ and $f(a) = f(b)$ then it must be the case that:

$$1 < a < e \text{ and } e < b < \infty.$$

Since $2 < e < 3$ then if we also want a and b to be integers with $a < b$ and $f(a) = f(b)$ then it follows that $a = 2$ and $b \geq 3$. Thus we now want to solve:

$$\frac{\ln(b)}{b} = \frac{\ln(2)}{2} \text{ for } b \geq 3.$$

Next since $f(x)$ is decreasing on (e, ∞) we see that there is at most one solution of $\frac{\ln(b)}{b} = \frac{\ln(2)}{2}$. Finally observe that

$$\frac{\ln(4)}{4} = \frac{2\ln(2)}{4} = \frac{\ln(2)}{2} = f(a)$$

and so we see that $b = 4$. □