SOLUTION FOR NOVEMBER 2020

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PROBLEM: Find all Pythagorean triples where the area of the triangle is equal to its perimeter.

SOLUTION: $(5, 12, 13)$ and $(6, 8, 10)$.

Proof: Suppose (a, b, c) is a Pythagorean triple (so a, b, c are positive integers) and the perimeter of the triangle equals the area so $c^2 = a^2 + b^2$ and $a + b + c = \frac{1}{2}ab$. Thus:

$$2c = ab - 2(a + b)$$

hence:

$$4(a^2 + b^2) = 4c^2 = a^2b^2 - 4ab(a + b) + 4(a^2 + 2ab + b^2)$$

therefore:

$$0 = ab(ab - 4(a + b) + 8).$$

Thus:

$$b = 4 + \frac{8}{a - 4}.$$

Since a, b are integers then we see that $a - 4$ divides evenly into 8 thus: $a - 4 = 1, 2, 4,$ or 8. Thus:

$$a = 5, 6, 8, \text{ or } 12.$$

Since $b = 4 + \frac{8}{a-4}$ and $c^2 = a^2 + b^2$ then we see that the possibilities are:

$$(5, 12, 13), (6, 8, 10), (8, 6, 10), \text{ and } (12, 5, 13).$$

Since the first and last set are the same and the second and third sets are the same we see we get the two solutions:

$$(5, 12, 13) \text{ and } (6, 8, 10).$$

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