SOLUTION FOR DECEMBER 2020

Correct solutions were submitted by:

Show that there are no rational points on the circle $x^2 + y^2 = 3$.

SOLUTION: We first notice that $(0, \pm\sqrt{3})$ and $(\pm\sqrt{3}, 0)$ are not rational points on $x^2+y^2 = 3$. Suppose now that $(m/n, r/s)$ is a rational point on the circle $x^2+y^2 = 3$ with $m \neq 0$ and $r \neq 0$ and m/n, r/s in lowest terms. Then $(\pm m/n, \pm r/s)$ are rational points on $x^2 + y^2 = 3$ and so without loss of generality we may assume $(m/n, r/s)$ is in the first quadrant and thus m, n, r, s are positive integers with m/n, r/s in lowest terms. Then:

$$\frac{m^2}{n^2} + \frac{r^2}{s^2} = 3 \tag{1}$$

and thus:

$$m^2 s^2 = n^2(3s^2 - r^2).$$

Thus n^2 divides the right-hand side and so n^2 divides the left-hand side and since m/n is in lowest terms then n^2 divides s^2 and so n divides s. Similarly s^2 divides into the left-hand side and so s^2 divides the right-hand side. Now s^2 does not divide into $3s^2 - r^2$ because if it did then s would divide into r but this cannot happen because r/s is in lowest terms. Thus s^2 divides into n^2 and so s divides into n. Since we also know n divides into s it follows then that $n = s$. Therefore we now see that (1) becomes:

$$m^2 + r^2 = 3n^2. \tag{2}$$

Next we claim that m or r is divisible by 3. If not then $m = 3k+1$ or $m = 3k+2$ and similarly $r = 3q + 1$ or $r = 3q + 2$. If $m = 3k + 1$ and $r = 3q + 1$ then substituting into (2) gives:

$$3(3k^2 + 2k + 3q^2 + 2q) + 2 = 3n^2$$

but the right-hand side is divisible by 3 and the left-hand side is not. Similarly we get a contradiction if $m = 3k + 1$ and $n = 3q + 2$, or $m = 3k + 2$ and $n = 3q + 1$, or $m = 3k + 2$ and $n = 3q + 2$. Thus at least one of m or r is divisible by 3. Let us now suppose that m is divisible by 3 so $m = 3m_1$. Substituting into (2) gives $r^2 = 3(n^2 - 3m_1^2)$ and so we see that r is also divisible by 3. (Similarly if r is divisible by 3 then it follows that m is divisible by 3). Thus $m = 3m_1$ and $r = 3r_1$. Substituting into (2) gives:

$$3(m_1^2 + n_1^2) = n^2.$$

Thus 3 divides evenly into n and we also know 3 divides evenly into m which contradicts that m/n is in lowest terms. Thus there are no rational points on $x^2 + y^2 = 3$.