SOLUTION FOR MARCH 2020

Correct solutions were submitted by:

Let $m < n < p$ be positive integers and suppose N is an integer such that:

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = N. \tag{1}$$

Prove that there is only one solution and find the solution.

SOLUTION:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

is the only solution.

Proof: Since $m < n < p$ then $m \geq 1$, $n \geq 2$, and $p \geq 3$ thus $N = \frac{1}{m}+\frac{1}{n}+\frac{1}{p} \leq 1+\frac{1}{2}+\frac{1}{3} = \frac{11}{6} < 2$. Since N must be an integer then we see $N = 1$. This then implies $m > 1$ so $m \geq 2$.

We next claim that $m = 2$. So suppose not. Then $m \geq 3$, $n \geq 4$, $m \geq 5$ so that $1 = \frac{1}{m} + \frac{1}{n} + \frac{1}{p} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$ which is impossible. Thus $m = 2$ and so (1) becomes:

$$\frac{1}{n} + \frac{1}{p} = \frac{1}{2}.$$

Since $p > n > m = 2$ then we see $n \geq 3$.

We next claim that $n = 3$. So suppose not. Then $n \geq 4$ and $p \geq 5$ so $\frac{1}{2} = \frac{1}{n} + \frac{1}{p} \leq \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$ which is impossible. Therefore $n = 3$ from which it follows that $p = 6$. $\square$