SOLUTION FOR OCTOBER 2021

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Let $a > 0$ and $d > 0$. Let:

$$A_n = \frac{1}{n}\sum_{k=0}^{n-1}(a+kd)$$

and:

$$G_n = \left(a(a+d)(a+2d)\cdots(a+(n-1)d)\right)^{1/n}. \tag{1}$$

Determine:

$$\lim_{n\to\infty}\frac{G_n}{A_n}.$$

SOLUTION:

$$\lim_{n\to\infty}\frac{G_n}{A_n} = \frac{2}{e}.$$

Proof: First since $\frac{1}{n}\sum\limits_{k=0}^{n-1} k = \frac{n-1}{2}$ then we see that $A_n = a + \frac{(n-1)d}{2}$. Next we rewrite:

$$\frac{G_n}{A_n} = \left(\frac{a}{a+\frac{(n-1)d}{2}}\cdot\frac{a+d}{a+\frac{(n-1)d}{2}}\cdots\frac{a+(n-1)d}{a+\frac{(n-1)d}{2}}\right)^{1/n}.$$

Taking ln gives:

$$\ln\left(\frac{G_n}{A_n}\right) = \frac{1}{n}\sum_{k=0}^{n-1}\ln\left(\frac{a+kd}{a+\frac{(n-1)d}{2}}\right).$$

Denoting $x_k = \frac{a+kd}{a+\frac{(n-1)d}{2}}$ then notice that $\Delta x_k = x_k - x_{k-1} = \frac{d}{a+\frac{(n-1)d}{2}}$ so we may rewrite this as:

$$\ln\left(\frac{G_n}{A_n}\right) = \left(\frac{a+\frac{(n-1)d}{2}}{d}\cdot\frac{1}{n}\right)\sum_{k=0}^{n-1}\ln(x_k)\Delta x_k.$$

The limit of the product of the first terms is $\frac{1}{2}$ and the summation is a Riemann sum for $\ln(x)$ on $[0,2]$. Thus:

$$\int_0^2 \ln(x)\,dx = \lim_{a\to 0^+}\int_a^2 \ln(x)\,dx = \lim_{a\to 0^+}(x\ln(x)-x)\big|_a^2 = \lim_{a\to 0^+}(2\ln(2)-2-a\ln(a)+a) = 2\ln(2)-2.$$

Thus:

$$\lim_{n\to\infty}\ln\left(\frac{G_n}{A_n}\right) = \frac{1}{2}(2\ln(2)-2) = \ln(2)-1.$$

Therefore:

$$\lim_{n\to\infty} \frac{G_n}{A_n} = \frac{2}{e}.$$

□