SOLUTION FOR NOVEMBER 2021

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Let $0 < b_1 < a_1$ and let:

$$a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \frac{2a_n b_n}{a_n + b_n}. \tag{1}$$

Show that:

$$0 < b_n < b_{n+1} < a_{n+1} < a_n$$

and determine:

$$\lim_{n\to\infty} a_n \text{ and } \lim_{n\to\infty} b_n.$$

SOLUTION:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sqrt{a_1 b_1}.$$

PROOF: Since $0 < b_1 < a_1$ notice then that $a_2 > 0$ and $b_2 > 0$. It then follows by induction that $a_n > 0$ and $b_n > 0$ for all n. Next observe that:

$$a_{n+1}b_{n+1} = a_n b_n, \tag{2}$$

$$a_{n+1} - a_n = \frac{b_n - a_n}{2}, \tag{3}$$

and:

$$b_{n+1} - b_n = \frac{b_n(a_n - b_n)}{a_n + b_n}. \tag{4}$$

Also:

$$a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{2(a_n + b_n)}. \tag{5}$$

Since $a_1 - b_1 > 0$ then it follows from (5) and by induction that $a_{n+1} - b_{n+1} > 0$ therefore $a_n > b_n$ for all n. Substituting this into (3)-(4) gives that $b_n < b_{n+1} < a_{n+1} < a_n$.

Therefore the b_n are increasing and bounded above by a_1 whereas the a_n are decreasing and bounded below by $b_1 > 0$. Thus there exist A, B with $0 < B \leq A$ such that:

$$\lim_{n\to\infty} a_n = A$$

and:

$$\lim_{n\to\infty} b_n = B.$$

Taking limits in (3) gives:

$$0 = A - A = \frac{B - A}{2}$$

and thus:

$$B = A.$$

Finally taking limits in (1) we see:

$$AB = A^2 = a_1 b_1.$$

Thus we see that:

$$B = A = \sqrt{a_1 b_1}.$$

$\square$