SOLUTION FOR DECEMBER 2021

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Determine:

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}. \tag{1}$$

SOLUTION:

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3}.$$

PROOF: Here we repeatedly use the identities:

$$(x-1)^3 = (x-1)(x^2+x+1)$$

and

$$(x+1)^3 = (x+1)(x^2-x+1).$$

Using these identities one can show that:

$$\left(\frac{k^3-1}{k^3+1}\right)\left(\frac{(k+1)^3-1}{(k+1)^3+1}\right) = \left(\frac{(k-1)k}{k^2-k+1}\right)\left(\frac{(k+1)^2+(k+1)+1}{(k+1)(k+2)}\right).$$

Similarly:

$$\left(\frac{k^3-1}{k^3+1}\right)\left(\frac{(k+1)^3-1}{(k+1)^3+1}\right)\left(\frac{(k+2)^3-1}{(k+2)^3+1}\right) = \left(\frac{(k-1)k}{k^2-k+1}\right)\left(\frac{(k+2)^2+(k+2)+1}{(k+2)(k+3)}\right).$$

One can then show by induction that:

$$\left(\frac{k^3-1}{k^3+1}\right)\left(\frac{(k+1)^3-1}{(k+1)^3+1}\right)\cdots\left(\frac{(k+n)^3-1}{(k+n)^3+1}\right) = \left(\frac{(k-1)k}{k^2-k+1}\right)\left(\frac{(k+n)^2+(k+n)+1}{(k+n)(k+n+1)}\right).$$

Letting $k=2$ gives:

$$\left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)\cdots\left(\frac{(2+n)^3-1}{(2+n)^3+1}\right) = \frac{2}{3}\left(\frac{(2+n)^2+(2+n)+1}{(2+n)(3+n)}\right).$$

Denoting $m=2+n$ gives:

$$\left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)\cdots\left(\frac{m^3-1}{m^3+1}\right) = \frac{2}{3}\left(\frac{m^2+m+1}{m(m+1)}\right).$$

Then as $m \to \infty$ the right-hand side goes to

$$\frac{2}{3}\cdot 1 = \frac{2}{3}.$$

□