

## SOLUTION FOR SEPTEMBER 2021

Correct solutions were submitted by:

Divya Darji

Michael Holland

Michalis Paizanis

Eric Peng

Josiah Sweatt

Determine:

$$\lim_{n \rightarrow \infty} n \sin(2\pi n!).$$

Hint:  $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ .

**SOLUTION:**

$$\lim_{n \rightarrow \infty} n \sin(2\pi n!) = 2\pi.$$

**Proof:** Using the hint we see that:

$$n!e = \text{integer} + \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \dots = \text{integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots$$

Thus:

$$2\pi n!e = 2\pi(\text{integer}) + \frac{2\pi}{n+1} + \frac{2\pi}{(n+1)(n+2)} + \dots$$

and since  $\sin(x)$  is  $2\pi$  periodic we see that:

$$\sin(2\pi n!e) = \sin\left(\frac{2\pi}{n+1} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right)\right) = \sin\left(\frac{2\pi}{n+1} (1 + \delta)\right)$$

where  $0 < \delta = \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots < \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \dots = \frac{1}{n+1}$ .

Next we recall:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

Thus for  $x$  sufficiently small we have:

$$1 - \epsilon < \frac{\sin(x)}{x} < 1 + \epsilon.$$

So for  $x = \frac{2\pi}{n+1}(1 + \delta)$  we have:

$$1 - \epsilon < \frac{\sin\left(\frac{2\pi}{n+1}(1 + \delta)\right)}{\frac{2\pi}{n+1}(1 + \delta)} < 1 + \epsilon.$$

Thus:

$$n \sin(2\pi n!e) = 2\pi \left( \frac{\sin\left(\frac{2\pi}{n+1}(1+\delta)\right)}{\frac{2\pi}{n}} \right) = 2\pi \left( \frac{\sin\left(\frac{2\pi}{n+1}(1+\delta)\right)}{\frac{2\pi}{n+1}(1+\delta)} \right) \frac{n(1+\delta)}{n+1}.$$

Also since  $0 < \delta < \frac{1}{n+1}$  we have:

$$\left(1 - \frac{1}{n+1}\right) < \frac{n(1+\delta)}{n+1} < 1.$$

Thus for  $n$  sufficiently large we have:

$$2\pi\left(1 - \frac{1}{n+1}\right)(1-\epsilon) < n \sin(2\pi n!e) < 2\pi(1+\epsilon).$$

Thus:

$$\lim_{n \rightarrow \infty} n \sin(2\pi n!e) = 2\pi.$$

□