SOLUTION FOR APRIL 2016

Let $x > 0$ and $y > 0$. Let:

$$f(x, y) = \int_0^\infty \frac{1}{(1 + x^2 t^2)(1 + y^2 t^2)} \, dt.$$

Prove that:

$$f(x, y) = \frac{\pi}{2(x + y)}$$

and then calculate:

$$\int_0^1 \left( \int_0^1 f(x, y) \, dx \right) \, dy$$

and determine:

$$\int_0^\infty \frac{(\arctan t)^2}{t^2} \, dt.$$ 

SOLUTION: Using partial fractions we see that:

$$(x^2 - y^2) f(x, y) = \int_0^\infty \frac{x^2 - y^2}{(1 + x^2 t^2)(1 + y^2 t^2)} \, dt = \int_0^\infty \frac{1}{1 + x^2 t^2} \, dt - \int_0^\infty \frac{1}{1 + y^2 t^2} \, dt$$

$$= (x \tan^{-1}(xt) - y \tan^{-1}(yt)) \Big|_0^\infty = \frac{\pi}{2}(x - y).$$

Thus for $x \neq y$ we see:

$$f(x, y) = \frac{\pi}{2(x + y)}.$$

We also notice that $f(x, y)$ is a continuous function and therefore it follows that:

$$f(x, y) = \frac{\pi}{2(x + y)}$$

for all $x > 0, y > 0$.

Next integrating in $x$ we get:

$$\int_0^1 \frac{\pi}{2(x + y)} \, dx = \int_0^1 \int_0^\infty \frac{1}{(1 + x^2 t^2)(1 + y^2 t^2)} \, dt \, dx.$$ 

We now interchange the order of integration on the right-hand side to obtain:

$$\frac{\pi}{2} [\ln(1 + y) - \ln(y)] = \int_0^\infty \int_0^1 \frac{1}{(1 + x^2 t^2)(1 + y^2 t^2)} \, dx \, dt$$

$$= \int_0^\infty \tan^{-1}(xt) \, \frac{1}{t} \left[ \frac{1}{1 + y^2 t^2} \right] \, dt = \int_0^\infty \tan^{-1}(t) \, \frac{1}{1 + y^2 t^2} \, dt.$$ 

Then integrating in $y$ and changing the order of integration again gives:

$$\int_0^1 \frac{\pi}{2} [\ln(1 + y) - \ln(y)] \, dy = \int_0^1 \int_0^\infty \frac{\tan^{-1}(t)}{t} \, \frac{1}{1 + y^2 t^2} \, dt \, dy.$$
\[
\int_0^\infty \int_0^1 \frac{\tan^{-1}(t)}{t} \frac{1}{1+y^2t^2} \, dy \, dt = \int_0^\infty \frac{(\tan^{-1}(t))^2}{t^2} \, dt.
\]

Finally working out the integral on the left we obtain:

\[
\int_0^1 \frac{\pi}{2} [\ln(1+y) - \ln(y)] \, dy = \frac{\pi}{2} [(1 + y) \ln(1 + y) - y \ln(y)] \bigg|_{0}^{1} = \frac{\pi}{2} 2 \ln(2) = \pi \ln(2).
\]

Thus,

\[
\int_0^\infty \frac{(\tan^{-1}(t))^2}{t^2} \, dt = \pi \ln(2).
\]