SOLUTION FOR APRIL 2016

Let $x > 0$ and $y > 0$. Let:

$$f(x,y) = \int_0^\infty \frac{1}{(1+x^2t^2)(1+y^2t^2)}\,dt.$$

Prove that:

$$f(x,y) = \frac{\pi}{2(x+y)}$$

and then calculate:

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$$

and determine:

$$\int_0^\infty \frac{(\arctan t)^2}{t^2}\,dt.$$

SOLUTION: Using partial fractions we see that:

$$(x^2-y^2)f(x,y) = \int_0^\infty \frac{x^2-y^2}{(1+x^2t^2)(1+y^2t^2)}\,dt = \int_0^\infty \frac{1}{1+x^2t^2}\,dt - \int_0^\infty \frac{1}{1+y^2t^2}\,dt$$

$$= \left(x\tan^{-1}(xt) - y\tan^{-1}(yt) \right)|_0^\infty = \frac{\pi}{2}(x-y).$$

Thus for $x \neq y$ we see:

$$f(x,y) = \frac{\pi}{2(x+y)}.$$

We also notice that $f(x,y)$ is a continuous function and therefore it follows that:

$$f(x,y) = \frac{\pi}{2(x+y)} \text{ for all } x > 0, y > 0.$$

Next integrating in x we get:

$$\int_0^1 \frac{\pi}{2(x+y)}\,dx = \int_0^1 \int_0^\infty \frac{1}{(1+x^2t^2)(1+y^2t^2)}\,dt\,dx.$$

We now interchange the order of integration on the right-hand side to obtain:

$$\frac{\pi}{2}[\ln(1+y) - \ln(y)] = \int_0^\infty \int_0^1 \frac{1}{(1+x^2t^2)(1+y^2t^2)}\,dx\,dt$$

$$= \int_0^\infty \frac{\tan^{-1}(xt)}{t}|_0^1 \left[\frac{1}{1+y^2t^2} \right] dt = \int_0^\infty \frac{\tan^{-1}(t)}{t} \frac{1}{1+y^2t^2}\,dt.$$

Then integrating in y and changing the order of integration again gives:

$$\int_0^1 \frac{\pi}{2}[\ln(1+y) - \ln(y)]\,dy = \int_0^1 \int_0^\infty \frac{\tan^{-1}(t)}{t} \frac{1}{1+y^2t^2}\,dt\,dy$$

$$= \int_0^\infty \int_0^1 \frac{\tan^{-1}(t)}{t} \frac{1}{1+y^2t^2} \, dy \, dt = \int_0^\infty \frac{(\tan^{-1}(t))^2}{t^2} \, dt.$$

Finally working out the integral on the left we obtain:

$$\int_0^1 \frac{\pi}{2}[\ln(1+y) - \ln(y)] \, dy = \frac{\pi}{2} \left[(1+y)\ln(1+y) - y\ln(y)\right] \Big|_0^1 = \frac{\pi}{2} 2\ln(2) = \pi \ln(2).$$

Thus,

$$\int_0^\infty \frac{(\tan^{-1}(t))^2}{t^2} \, dt = \pi \ln(2).$$