

## SOLUTION FOR SEPTEMBER 2016

Determine:

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}.$$

Note: The notation  $\prod_{i=1}^{2n} b_i$  means the product of these elements. That is  $\prod_{i=1}^{2n} b_i = b_1 b_2 \cdots b_{2n-1} b_{2n}$ .

**SOLUTION:**

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n} = 25e^{-4+2 \tan^{-1}(2)}$$

First we write:

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n} = L$$

and then take logs of both sides and simplify using rules of logs to obtain:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{2n} \ln(n^2 + i^2) - 4 \ln(n) = \ln(L).$$

Next:

$$\ln(n^2 + i^2) = \ln\left(n^2 \left(1 + \frac{i^2}{n^2}\right)\right) = 2 \ln(n) + \ln\left(1 + \frac{i^2}{n^2}\right)$$

and therefore:

$$\frac{1}{n} \sum_{i=1}^{2n} \ln(n^2 + i^2) - 4 \ln(n) = \frac{1}{n} \sum_{i=1}^{2n} \left(2 \ln(n) + \ln\left(1 + \frac{i^2}{n^2}\right)\right) - 4 \ln(n) = \frac{1}{n} \sum_{i=1}^{2n} \ln\left(1 + \frac{i^2}{n^2}\right).$$

Next you might observe that this looks like a Riemann sum for the function  $\ln(1 + x^2)$  and in fact if we take a partition of  $[0, 2]$  by dividing this interval into  $2n$  subintervals and if we use the right-hand endpoint of each subinterval as a selection point then we see that we obtain:

$$\int_0^2 \ln(1 + x^2) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{2n} \ln\left(1 + \frac{i^2}{n^2}\right).$$

Therefore we just need to calculate  $\int_0^2 \ln(1 + x^2) dx$ . Integrating by parts gives:

$$\int_0^2 \ln(1 + x^2) dx = x \ln(1 + x^2) \Big|_0^2 - \int_0^2 \frac{2x^2}{1 + x^2} dx = 2 \ln(5) - \int_0^2 \frac{2x^2}{1 + x^2} dx$$

$$= 2 \ln(5) - \int_0^2 \frac{2x^2 + 2}{1 + x^2} dx + \int_0^2 \frac{2}{1 + x^2} dx = 2 \ln(5) - 4 + 2 \tan^{-1}(x) \Big|_0^2 = 2 \ln(5) - 4 + 2 \tan^{-1}(2).$$

Therefore:

$$\ln(L) = 2 \ln(5) - 4 + 2 \tan^{-1}(2)$$

and thus:

$$L = 25e^{-4+2 \tan^{-1}(2)}.$$