Sofic entropy theory

Lewis Bowen

Logic, Dynamics and their Interactions
with a celebration of the work of Dan Mauldin
University of North Texas
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Group actions

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Main Problem: Classify actions of G up to isomorphism.

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- The action $G \curvearrowright (K^G, \kappa^G) = (K, \kappa)^G$ is the *Bernoulli shift* over G with base measure κ.
- $H(K, \kappa) := -\sum_{k \in K'} \kappa(k) \log \left(\kappa(k)\right)$ if κ is supported on a countable set $K' \subset K$. Otherwise, $H(K, \kappa) := +\infty$.

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Factor maps

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Definition

Let $G \curvearrowright (X, \mu)$, $G \curvearrowright (Y, \nu)$ be two systems and $\phi : X \to Y$ a measurable map with $\phi_* \mu = \nu$, $\phi(gx) = g\phi(x)$ for a.e. $x \in X$ and all $g \in G$. Then ϕ is a *factor map* from $G \curvearrowright (X, \mu)$ to $G \curvearrowright (Y, \nu)$.

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(Sinai 1960s, Ornstein-Weiss 1980) If G is amenable then $G \curvearrowright (K,\kappa)^G$ factors onto $G \curvearrowright (L,\lambda)^G$ if and only if $H(K,\kappa) \geq H(L,\lambda)$.

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G amenable $\Rightarrow h(G \curvearrowright (\mathbb{Z}/n\mathbb{Z}, u_n)^G) = \log(n)$ (the full n-shift).

So the full 2-shift over G cannot factor onto the full 4-shift over G.

The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $\mathbb{F} = \langle a, b \rangle$ is the rank 2 free group then the full 2-shift over $\mathbb{F}$ factors onto the full 4-shift over $\mathbb{F}$.

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$$\phi(x)(g) = \Big(x(g) + x(ga), x(g) + x(gb)\Big).$$

Theorem (B. 2011)

If G contains a subgroup isomorphic to $\mathbb{F}$ then every nontrivial Bernoulli shift over G factors onto every other Bernoulli shift over G.

Factors between Bernoulli shifts

Theorem (Gaboriau-Lyons, 2009)

For every non-amenable group G there is some space (K, κ) *with* $H(K, \kappa) < \infty$ *and an essentially free action* $\mathbb{F} \curvearrowright (K, \kappa)^G$ *with* $\mathbb{F}$*-orbits contained in the G-orbits.*

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Theorem (Ball, 2005)

For every non-amenable group G there is some (K, κ) with $H(K, \kappa) < 0$ such that $G \curvearrowright (K, \kappa)^G$ factors onto every Bernoulli shift.

New Results: Sofic Groups

Theorem (B., 2010)

If G is a sofic group then Kolmogorov's direction holds. I.e., if $G \curvearrowright (K^G, \kappa^G)$ *is isomorphic to* $G \curvearrowright (L^G, \lambda^G)$ *then* $H(K, \kappa) = H(L, \lambda)$.

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History

1. L.B. 2010: free groups, sofic groups, finitely generated actions;
2. Kerr-Li 2011: topological and measure-theoretic sofic entropy; removed finite generation hypothesis
3. New equivalent definitions have been found by Kerr-Li, Kerr and Zhang.

Pseudo-metrics and separating sets

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Define pseudo-metrics ρ_2 and ρ_∞ on Y^n by:

$$\begin{aligned} \rho_2\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) &= \sqrt{\frac{1}{n}\sum_{i=1}^{n} \rho(x_i, y_i)^2}, \\ \rho_\infty\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) &:= \max_i \rho(x_i, y_i). \end{aligned}$$

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$$h_{top}(T, \rho) := \sup_{\epsilon>0} \limsup_{n\to\infty} n^{-1} \log N_\epsilon(\mathrm{Orb}(T, n), \rho_\infty)$$

ρ is dynamically generating if for any $x \neq y \in X$ there is some $n \in \mathbb{Z}$ such that $\rho(T^n x, T^n y) > 0$.

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Theorem (Rufus Bowen, 70's)

If ρ_1, ρ_2 are dynamically generating, then $h_{top}(T, \rho_1) = h_{top}(T, \rho_2)$. So $h_{top}(T) := h_{top}(T, \rho_1)$.

Topological entropy was defined earlier, using open covers, by Adler, Konheim and McAndrew (1965).

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Given a finite set $L \subset C(X)$ and $\delta > 0$, let

$$\mathrm{Orb}(T, L, \delta, n) := \Big\{(x, Tx, \ldots, T^{n-1}x) \in X^n : \ x \in X, \\ \left| \int f \, d\mu - \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \right| < \delta \ \forall f \in L \Big\}.$$

Measure-theoretic entropy

$$h_\mu(T,\rho) := \sup_{\epsilon>0} \inf_{L\subset C(X)} \inf_{\delta>0} \limsup_{n\to\infty} n^{-1} \log N_\epsilon(\mathrm{Orb}(T,L,\delta,n),\rho_\infty).$$

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Theorem

If ρ is dynamically generating, and T is ergodic, then $h_\mu(T,\rho) = h_\mu(T)$.

Sofic groups

Definition

G is sofic if there is a sequence $\sigma_i : G \to \mathrm{Sym}(d_i)$ of maps (which need not be homomorphisms!) such that for every $g, h \in G$,

$$\lim_{i\to\infty} d_i^{-1} \#\{p \in [d_i] : \ \sigma_i(g)\sigma_i(h)p = \sigma_i(gh)p\} = 1$$

and if $g \neq h$ then

$$\lim_{i\to\infty} d_i^{-1} \#\{p \in [d_i] : \ \sigma_i(g)p \neq \sigma_i(h)p\} = 1.$$

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- Amenable groups are sofic.
- (Gromov 1999, Weiss 2000, Elek-Szabo 2005) If G is sofic then G satisfies Gottshalk's surjunctivity conjecture, Connes embedding conjecture, the Determinant conjecture, Kaplansky's direct finiteness conjecture.
- Open: Is every countable group sofic?

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$\Sigma = \{\sigma_i\}$ be a sofic approximation to G where $\sigma_i : G \to \mathrm{Sym}(d_i)$,

Given $W \subset G$ finite, $\delta > 0$, let $\mathrm{Orb}(W, \delta, \sigma_i)$ be the set of all maps $\phi : [d_i] \to X$ such that

$$\rho_2(\phi \circ \sigma_i(w), w \circ \phi) < \delta, \ \forall w \in W.$$

These are approximate partial orbits.

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$$h_\Sigma(G \curvearrowright X, \rho) := \sup_{\epsilon>0} \inf_{W \subset G} \inf_{\delta>0} \limsup_{i\to\infty} d_i^{-1} \log N_\epsilon(\mathrm{Orb}(W,\delta,\sigma_i),\rho_\infty).$$

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Theorem (Kerr-Li, 2011)

If ρ_1, ρ_2 are dynamically generating, then $h_\Sigma(G \curvearrowright X, \rho_1) = h_\Sigma(G \curvearrowright X, \rho_2)$. So $h_\Sigma(G \curvearrowright X) := h_\Sigma(G \curvearrowright X, \rho_1)$. Moreover, if G is amenable, then this coincides with classical topological entropy.

Measure-entropy for sofic groups

Suppose $G \curvearrowright X$ preserves a probability measure μ on X.

For a finite set $L \subset C(X)$, let

$$\mathrm{Orb}(W, L, \delta, \sigma_i) := \Big\{\phi \in \mathrm{Orb}(W, \delta, \sigma_i) : \left| \int f \, d\mu - \frac{1}{d_i} \sum_{q \in d_i} f(\phi(q)) \right| < \delta \ \forall f \in L \Big\}.$$

Measure-entropy for sofic groups

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\begin{aligned}
& h_{\Sigma,\mu}(G \curvearrowright X, \rho) \\
:= \quad & \sup_{\epsilon>0} \inf_{W \subset G} \inf_{L \subset C(X)} \inf_{\delta>0} \limsup_{i\to\infty} d_i^{-1} \log N_\epsilon(\mathrm{Orb}(W, L, \delta, \sigma_i), \rho_\infty).
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If ρ_1, ρ_2 are dynamically generating, then $h_{\Sigma,\mu}(G \curvearrowright X, \rho_1) = h_{\Sigma,\mu}(G \curvearrowright X, \rho_2)$. So $h_{\Sigma,\mu}(G \curvearrowright X) := h_{\Sigma,\mu}(G \curvearrowright X, \rho_1)$. Moreover, if G is amenable, then this coincides with classical measure-entropy. Also, $h_{\Sigma,\kappa^G}(G \curvearrowright K^G) = H(K, \kappa)$.

The variational principle

Theorem (Kerr-Li, 2011)

Let $G \curvearrowright X$ be an action by homeomorphisms on a compact metrizable space. For any sofic approximation Σ,

$$h_\Sigma(G \curvearrowright X) = \sup_\mu h_{\Sigma,\mu}(G \curvearrowright X).$$

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Conjecture (Gottschalk's surjunctivity conjecture)

For any countable discrete group G, any finite set K and any continuous G-equivariant map $\phi : K^G \to K^G$, if ϕ is injective then must also be surjective.

This is known to hold for sofic groups: Gromov (1999), Weiss (2000), Kerr-Li (2011).

The annealed sofic entropy of free group actions

For each $n \geq 1$, let $\sigma_n : \mathbb{F} = \langle s_1, \ldots, s_r \rangle \to \mathrm{Sym}(n)$ be chosen uniformly at random.

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For each $n \geq 1$, let $\sigma_n : \mathbb{F} = \langle s_1, \ldots, s_r \rangle \to \mathrm{Sym}(n)$ be chosen uniformly at random.

Suppose $\mathbb{F} \curvearrowright X$ is an action by homeomorphisms on a compact metrizable space X with pseudo-metric ρ.
Define

$$\begin{aligned} & h^*_\mu(G \curvearrowright X, \rho) \\ := \; & \sup_{\epsilon>0} \inf_{W \subset G} \inf_{L \subset C(X)} \inf_{\delta>0} \limsup_{n\to\infty} n^{-1} \log \mathbb{E}[N_\epsilon(\mathrm{Orb}(W, L, \delta, \sigma_n), \rho_\infty)]. \end{aligned}$$

Free Groups: a special case

Given a finite partition $\mathcal{P}$ of X define

$$F_\mu(\mathcal{P}) := -(2r-1)H_\mu(\mathcal{P}) + \sum_{i=1}^{r} H_\mu(\mathcal{P} \vee s_i\mathcal{P});$$

$$f_\mu(\mathbb{F} \curvearrowright X, \mathcal{P}) := \inf_{W \subset \mathbb{F}} F_\mu\Big(\bigvee_{g \in W} g\mathcal{P}\Big).$$

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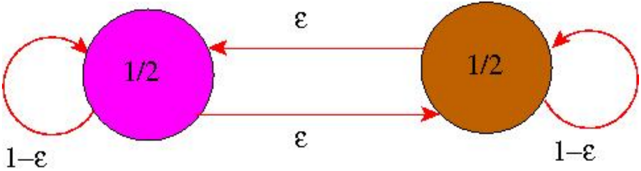
Theorem

If $\mathcal{P}$ is generating then $f_\mu(\mathcal{P}) = h^*_\mu(G \curvearrowright X)$.

A Markov chain example



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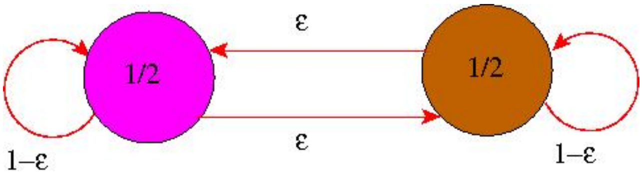
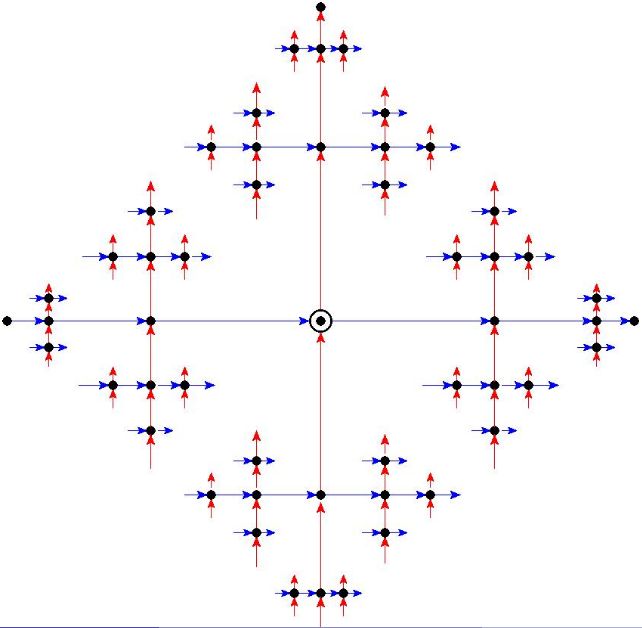
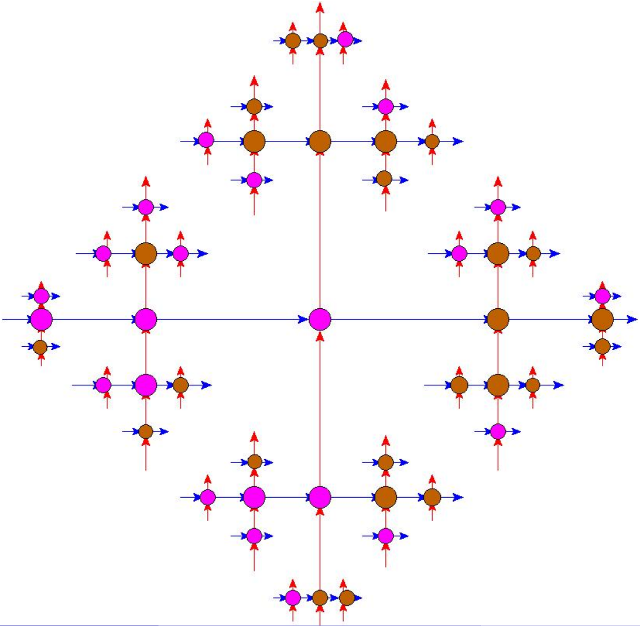


The Cayley graph



The Ising model



Example

$$h^*(\mathbb{F} \curvearrowright \{\text{magenta, brown}\}^{\mathbb{F}}) = -2\epsilon \log(\epsilon) - 2(1-\epsilon)\log(1-\epsilon) - \log(2).$$

Formulae

Let $\alpha : \mathbb{F} \to \mathrm{Aut}(X, \mu)$, $\beta : \mathbb{F} \to \mathrm{Aut}(Y, \nu)$ be actions.

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- direct product: $f(\alpha \times \beta) = f(\alpha) + f(\beta)$.
- n-1 factor maps: if $\pi : (X, \mu) \to (Y, \nu)$ is an n-to-1 factor map then $f(\beta) = f(\alpha) + (r-1)\log(n)$.

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$$\begin{aligned} f(\mathbb{F}, \mathcal{G}, \mathrm{Haar}_{\mathcal{G}}) &= f(\mathbb{F}, \mathcal{N}, \mathrm{Haar}_{\mathcal{N}}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, \mathrm{Haar}_{\mathcal{G}/\mathcal{N}}) \\ \log(2) &= -\log(2) + \log(4). \end{aligned}$$

Further Results & Open Questions

- There is a relative entropy theory for extensions of sofic groupoids.
- Is there an Ornstein theory for free groups ? e.g., does every system of positive entropy factor onto a Bernoulli shift?
- Sofic entropy of algebraic dynamical systems (L.B., Kerr-Li, L.B.-Li).
- Sofic topological pressure (Nhan-Phu Chung)
- Local sofic entropy theory (Zhang).
- Sofic dimension (Dykema-Kerr-Pichot) (analogous to Voiculescu's free entropy dimension).
- Sofic mean dimension (H. Li).
- l^p-version of von Neumann dimension for Banach space representations of sofic groups (Ben Hayes).