Equi-kneading of skew tent maps in the square

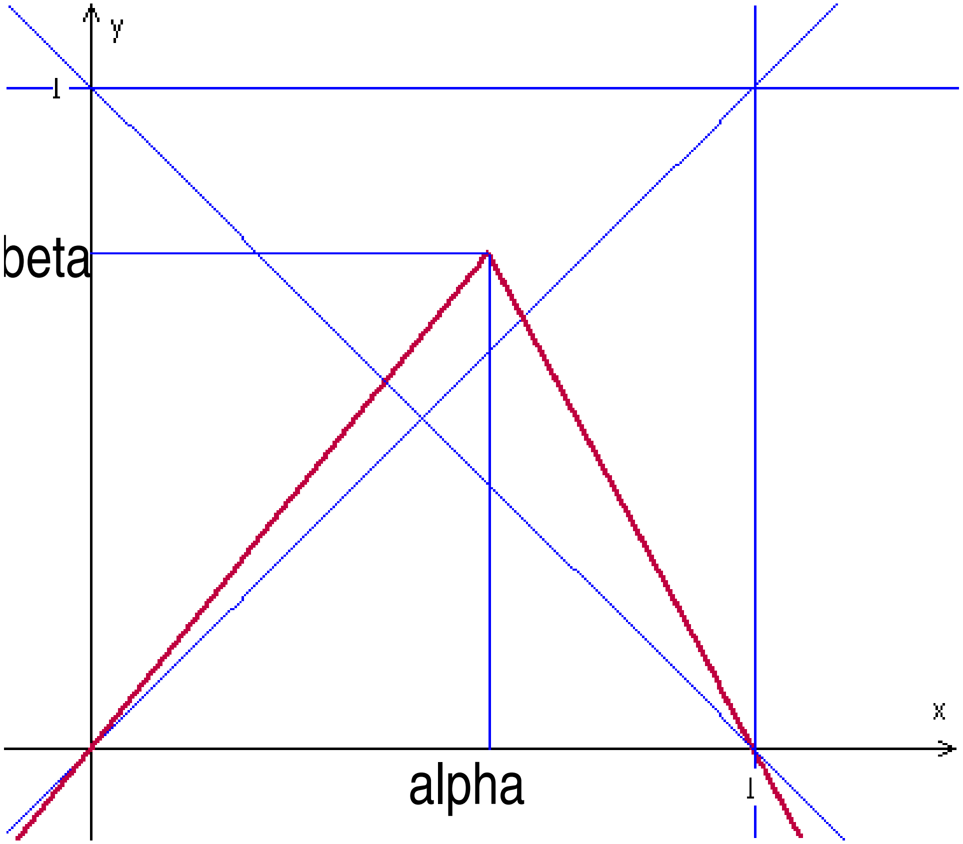
(work in progress)

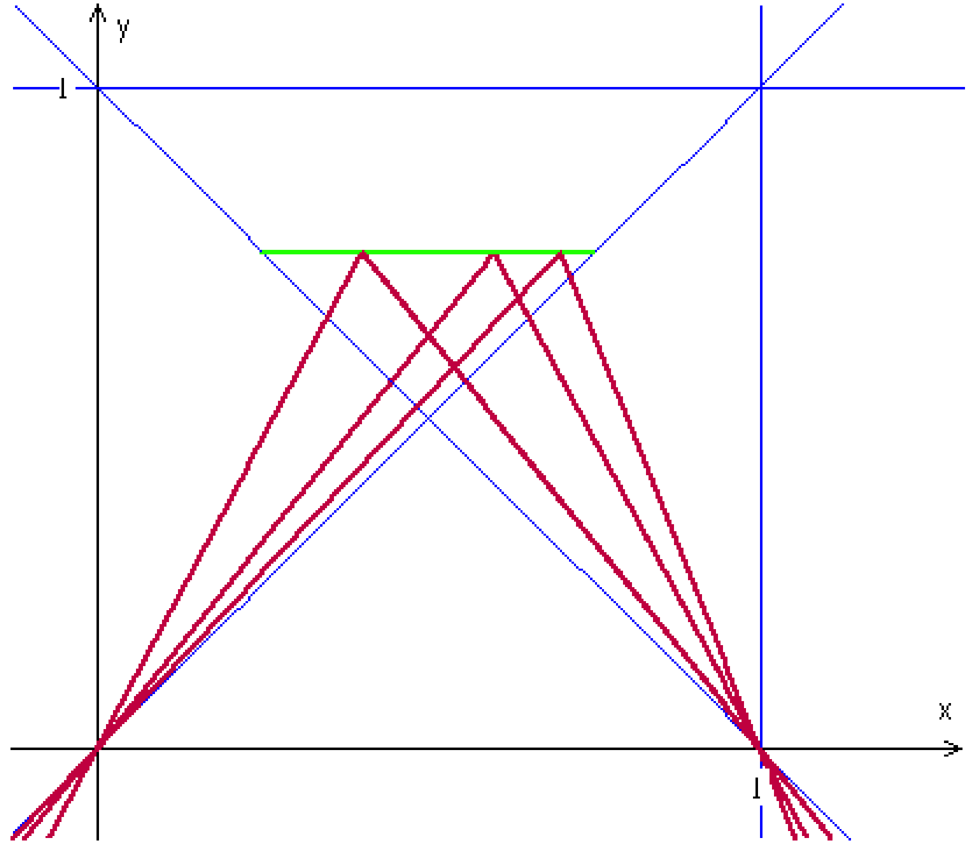
Zoltán Buczolich

Eötvös University, Budapest

www.cs.elte.hu/∼buczo

Joint work with: Gabriella Keszthelyi

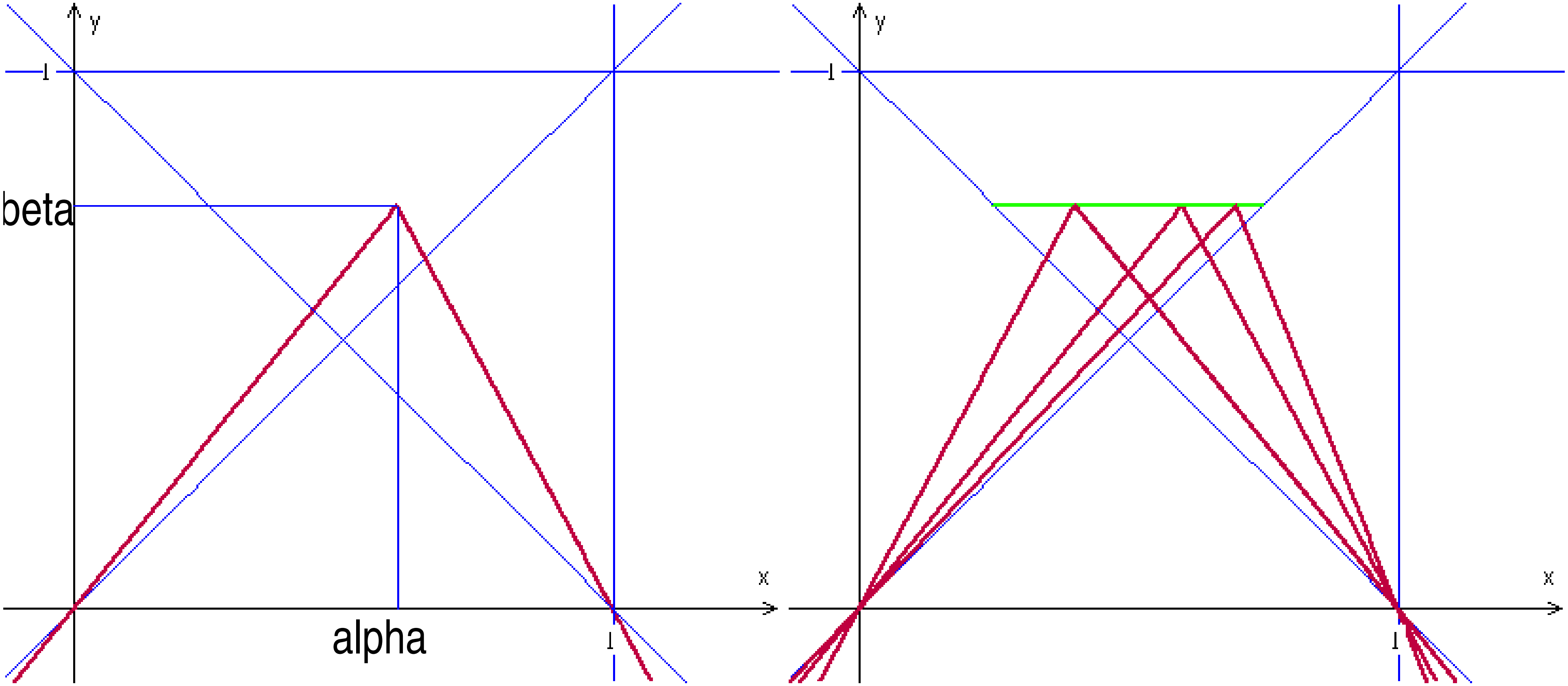




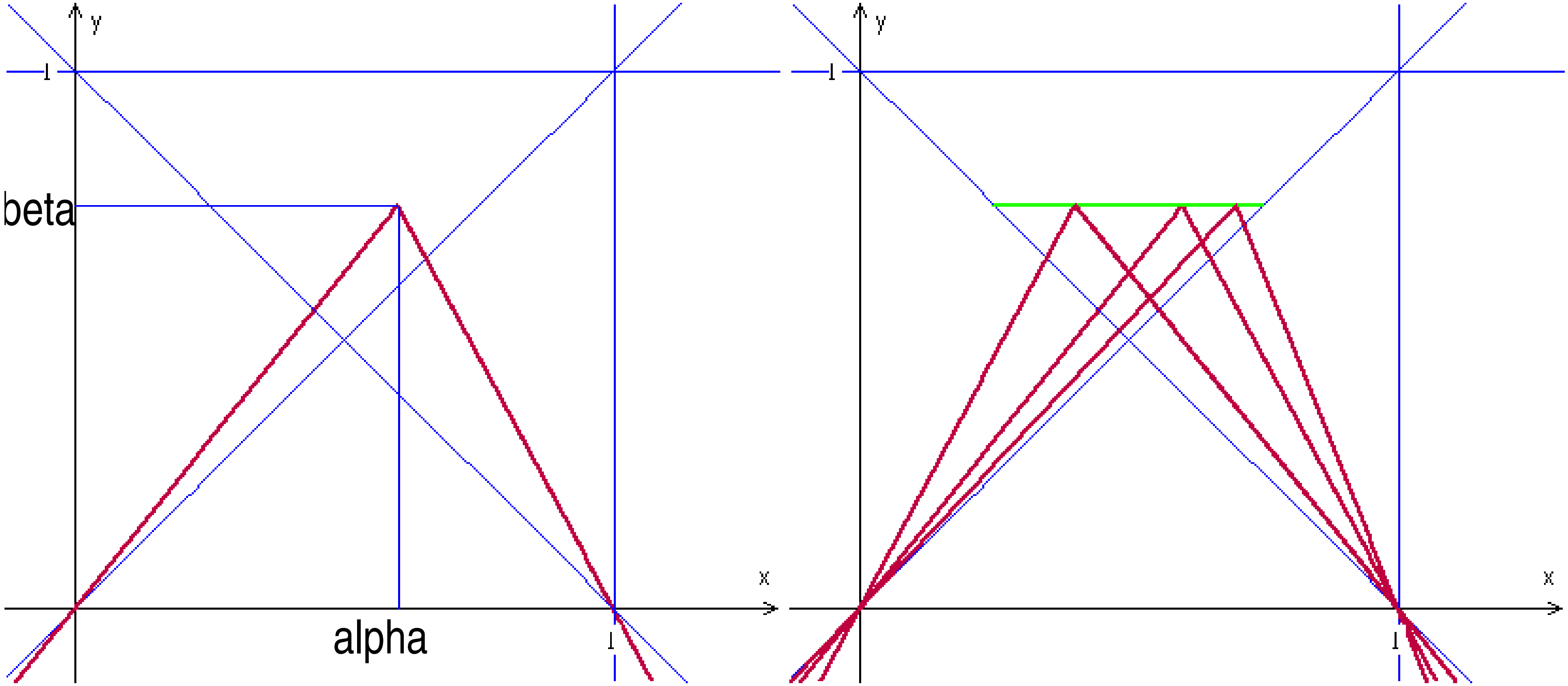
Consider a point (α, β) in the unit square $[0, 1]^2$.

Denote by $T_{\alpha,\beta}(x)$ the skew tent map

$$T_{\alpha,\beta}(x) = \begin{cases} \frac{\beta}{\alpha}x & \textit{if} \quad 0 \le x < \alpha \\ \frac{\beta}{1-\alpha}(1-x) & \textit{if} \quad \alpha < x \le 1. \end{cases}$$



The topological entropy of $T_{\alpha,\beta}$ is denoted by $h(\alpha,\beta)$.

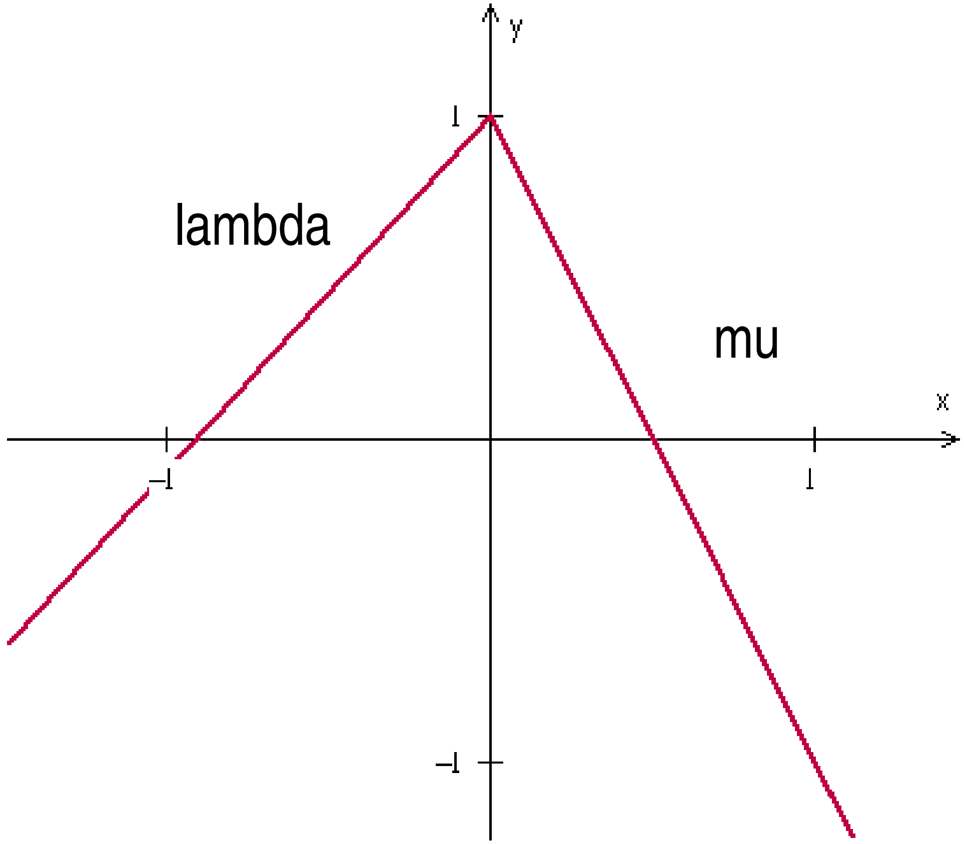


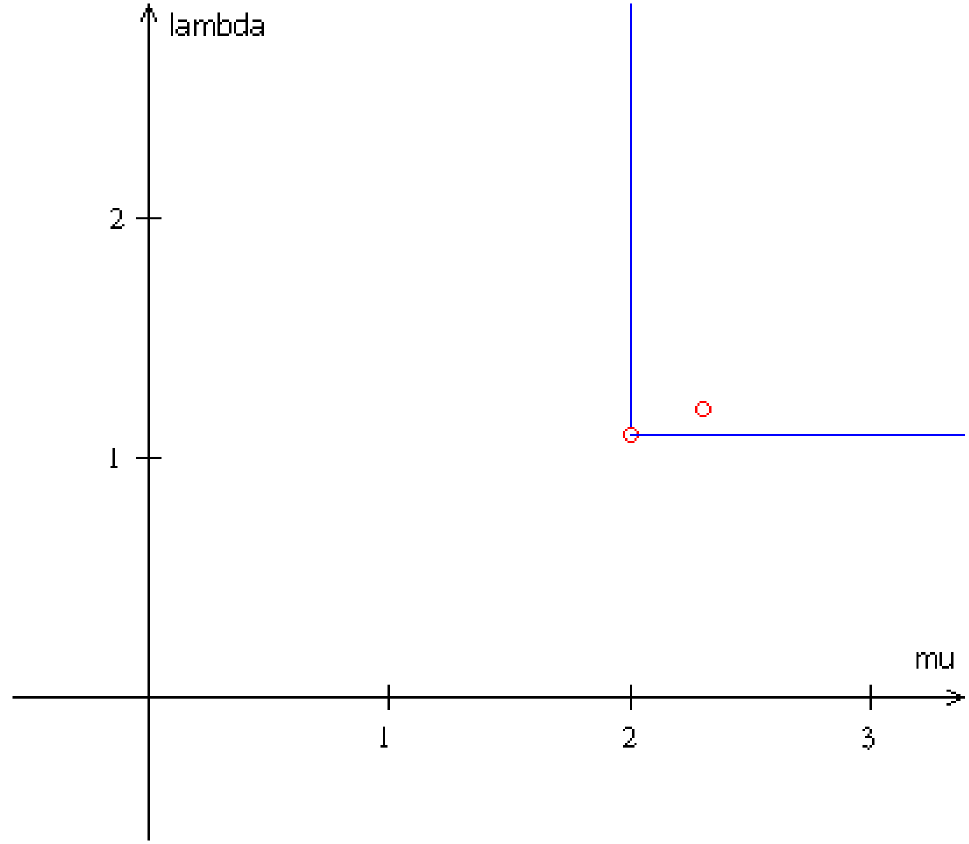
The topological entropy of $T_{\alpha,\beta}$ is denoted by $h(\alpha, \beta)$.

The starting point of our paper is the question about the behavior of the function $h(\alpha) = h(\alpha, \beta)$.

The answer to the question about the behavior of the function $g(\beta) = h(\alpha, \beta)$ with a fixed α is known.

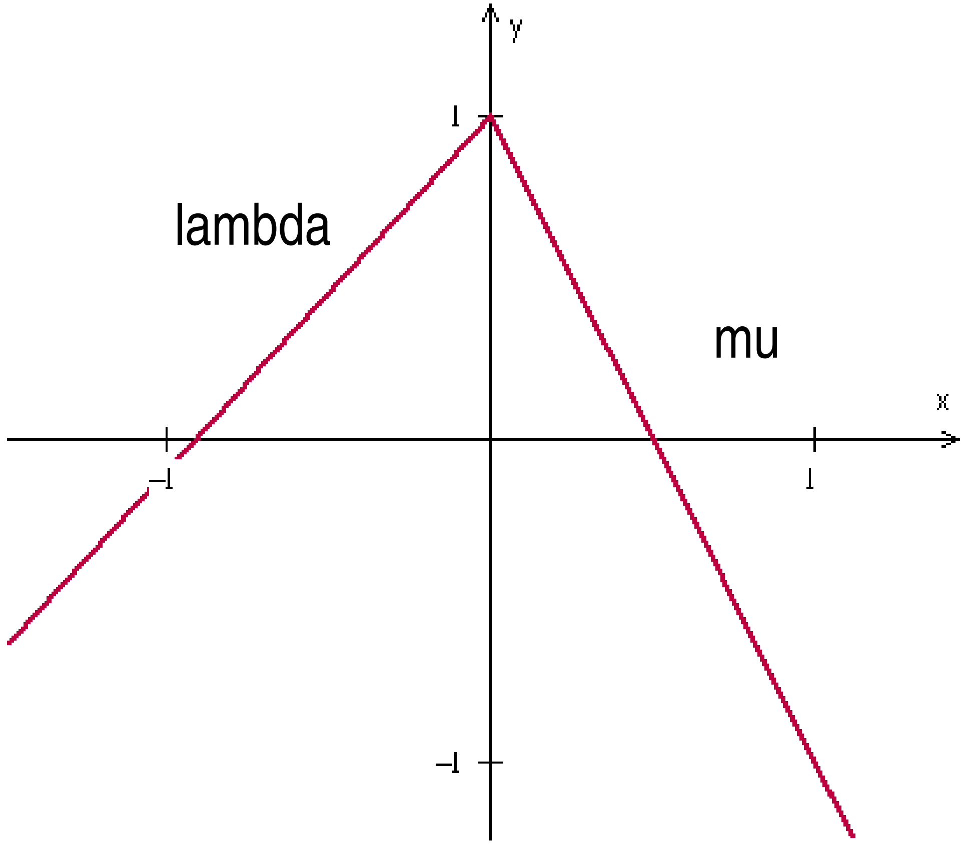
In the case of these $\beta's$ where the dynamics of $T_{\alpha,\beta}(x)$ is nontrivial the function $g(\beta)$ is monotone increasing.

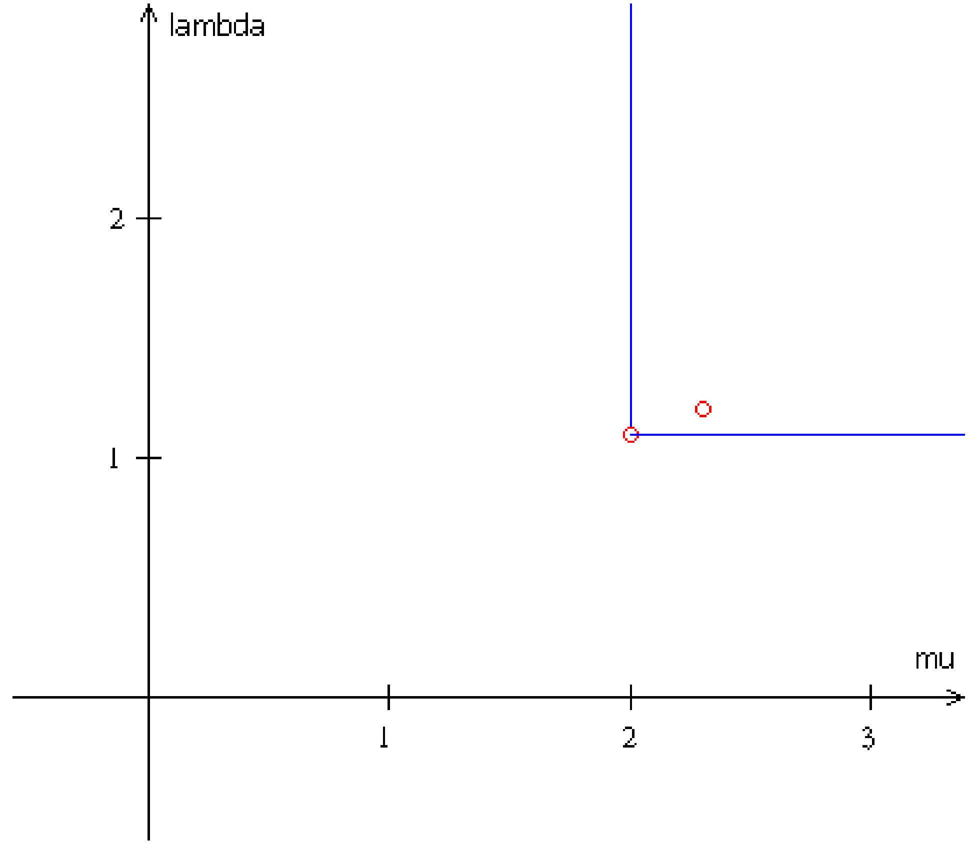




Skew tent maps and topological entropy were considered by M. Misiurewicz and E. Visinescu.

They used different parametrization: $F_{\lambda,\mu}(x) = \begin{cases} 1+\lambda x & \textit{if} \quad x \le 0 \\ 1-\mu x & \textit{if} \quad x \ge 0 \end{cases}$ on $\mathbb{R}$.





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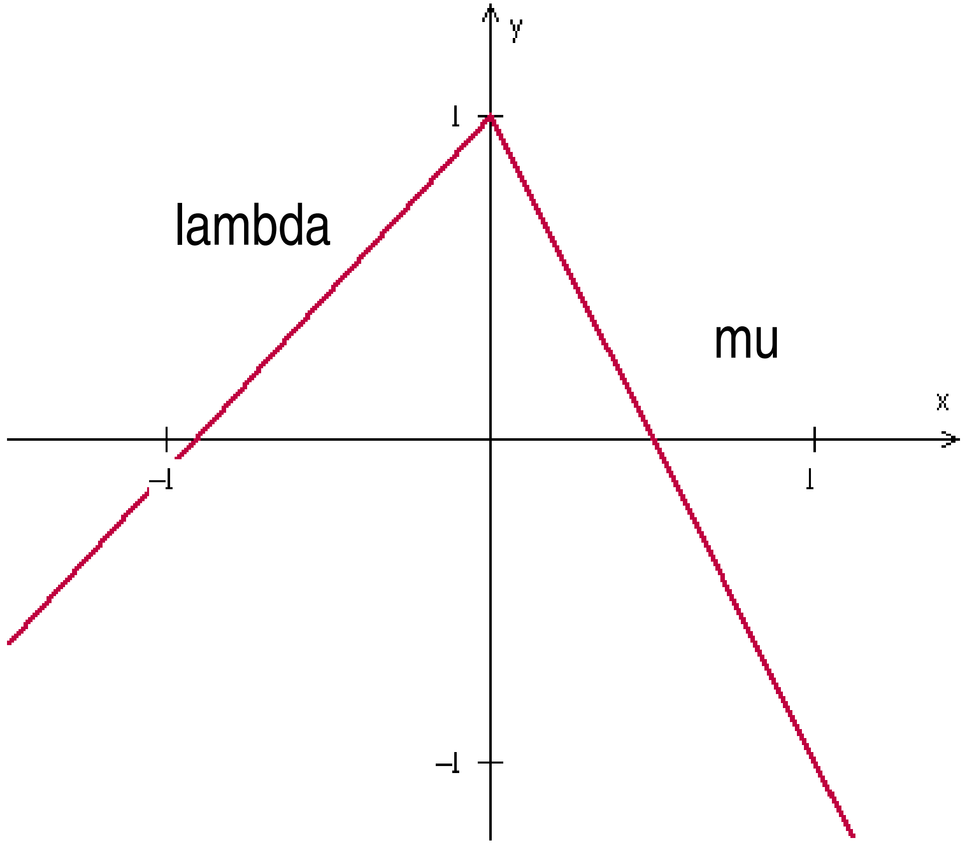
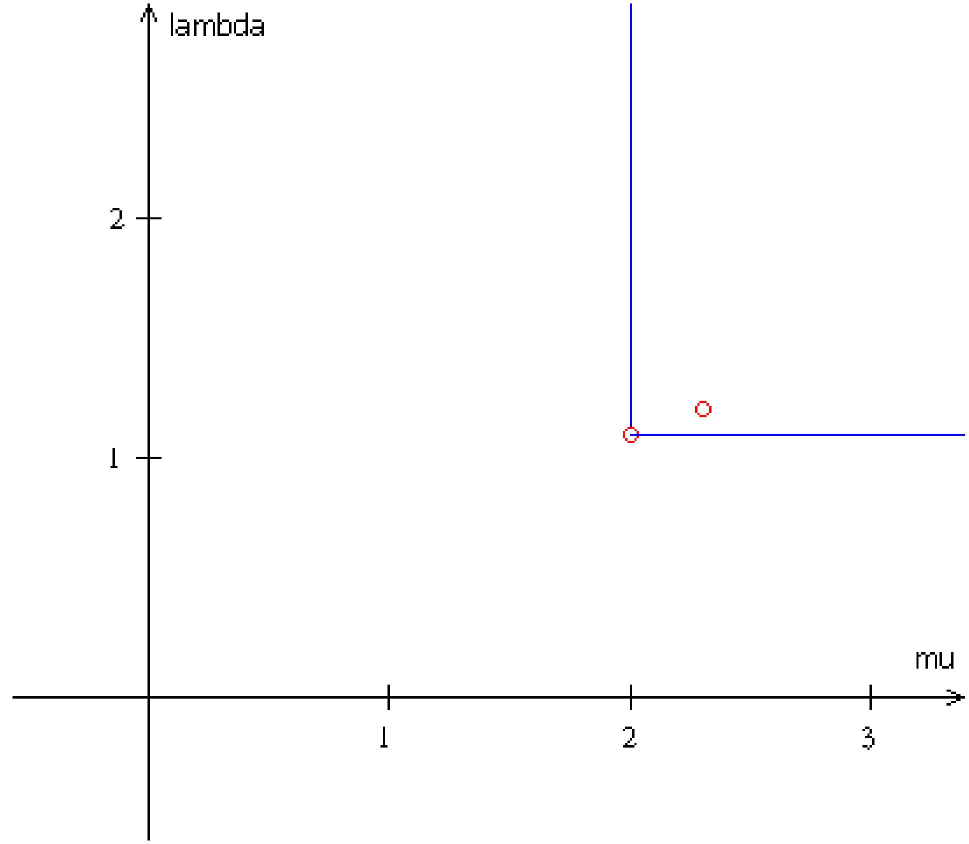
It is rather easy to see that if $\lambda = \frac{\beta}{\alpha}$, $\mu = \frac{\beta}{1-\alpha}$ and $h(x) = (\beta-\alpha)x + \alpha$, $h^{-1}(x) = \frac{x-\alpha}{\beta-\alpha}$ then

$$F_{\lambda,\mu}(x) = (h^{-1} \circ T_{\alpha,\beta} \circ h)(x),$$

provided that we extend the definition of $T_{\alpha,\beta}$ onto $\mathbb{R}$ int the obvious way.

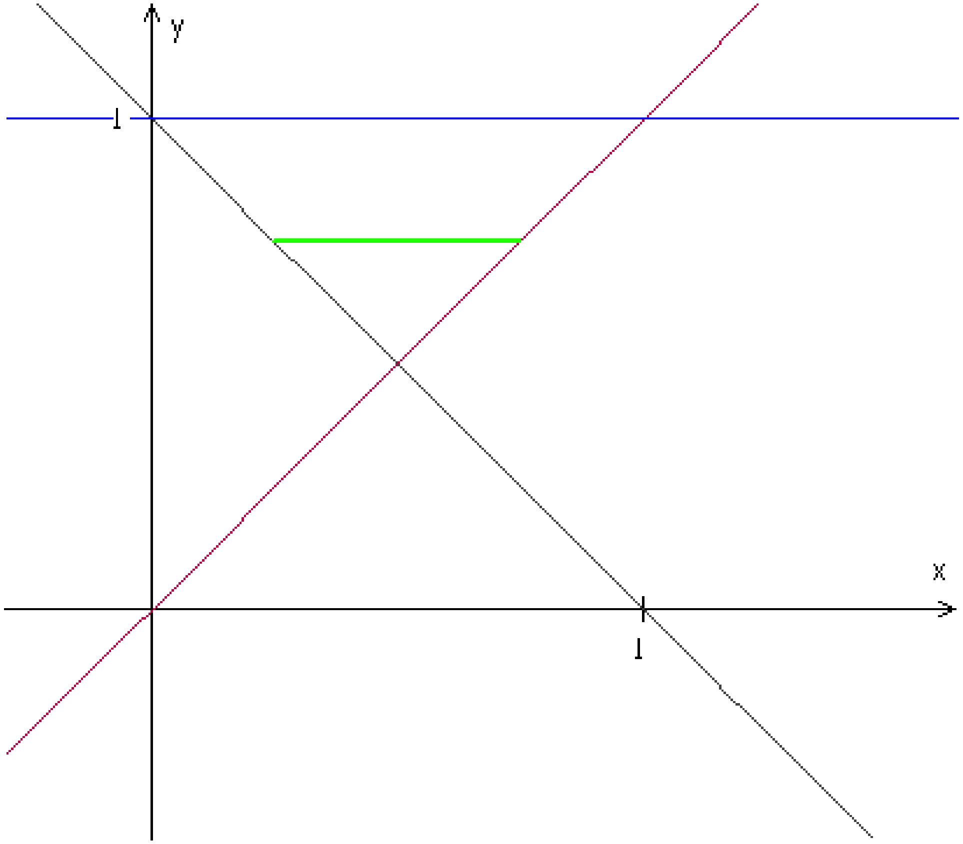


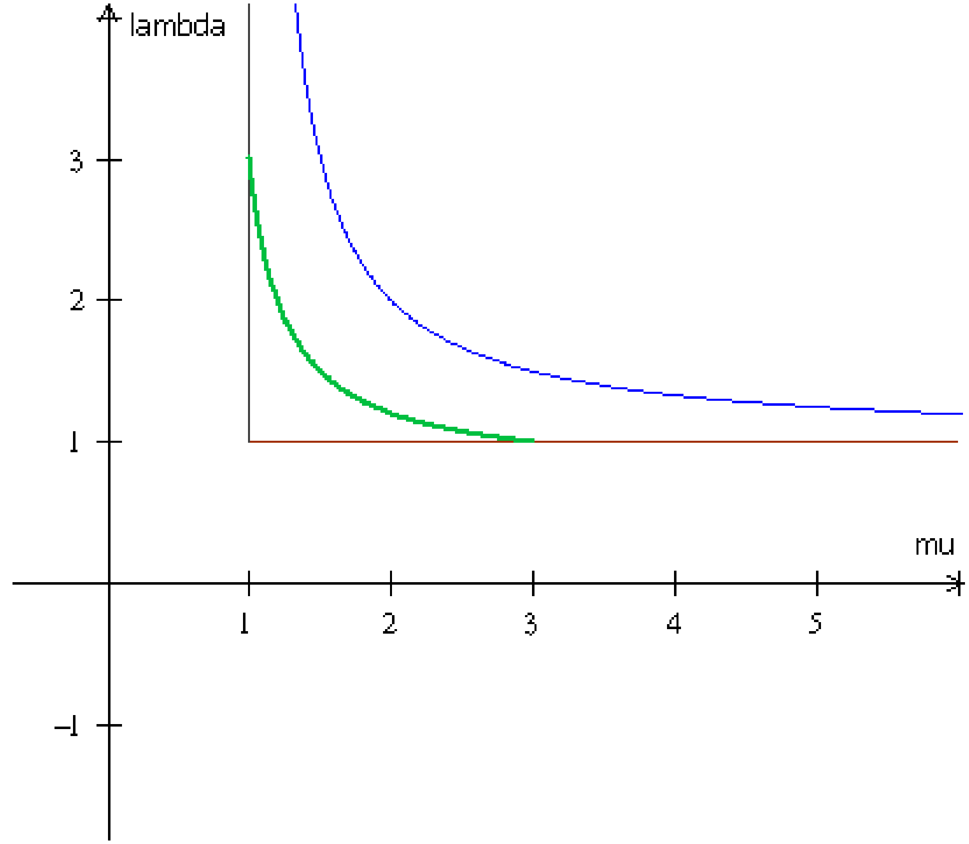


Results of M. Misiurewicz and E. Visinescu imply that if $\lambda' \geq \lambda$, $\mu' \geq \mu$ and at least one of these inequalities is sharp, then $h(F_{\lambda',\mu'}) > h(F_{\lambda,\mu})$ where $h(F_{\lambda,\mu})$ denotes the topological entropy of $F_{\lambda,\mu}$.

If β is fixed and α increases then $\lambda = \frac{\beta}{\alpha}$ decreases, while $\mu = \frac{\beta}{1-\alpha}$ increases.

This implies that the monotonicity result in [MV] is not giving an answer to our question.





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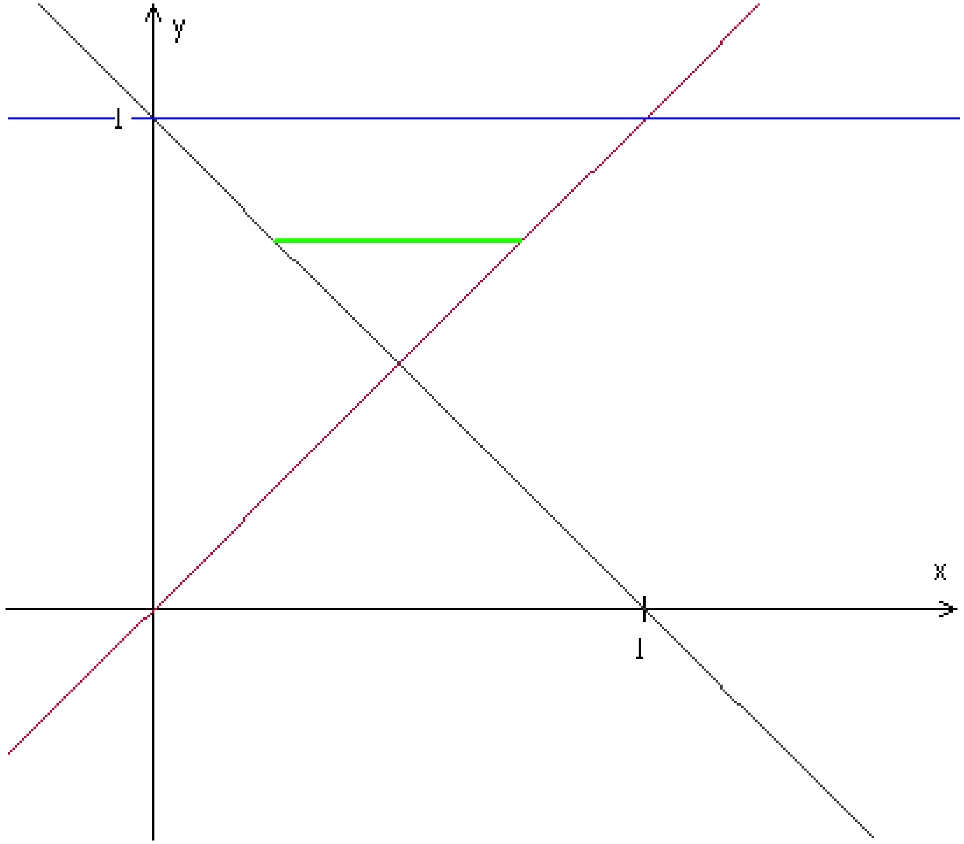
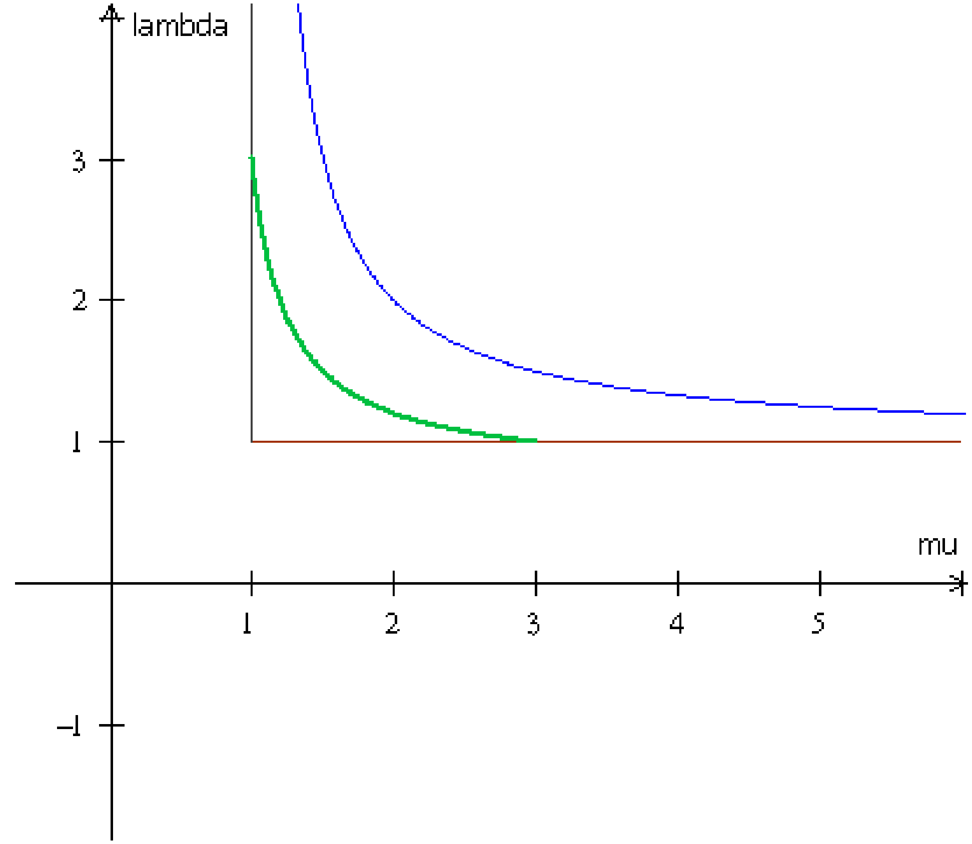
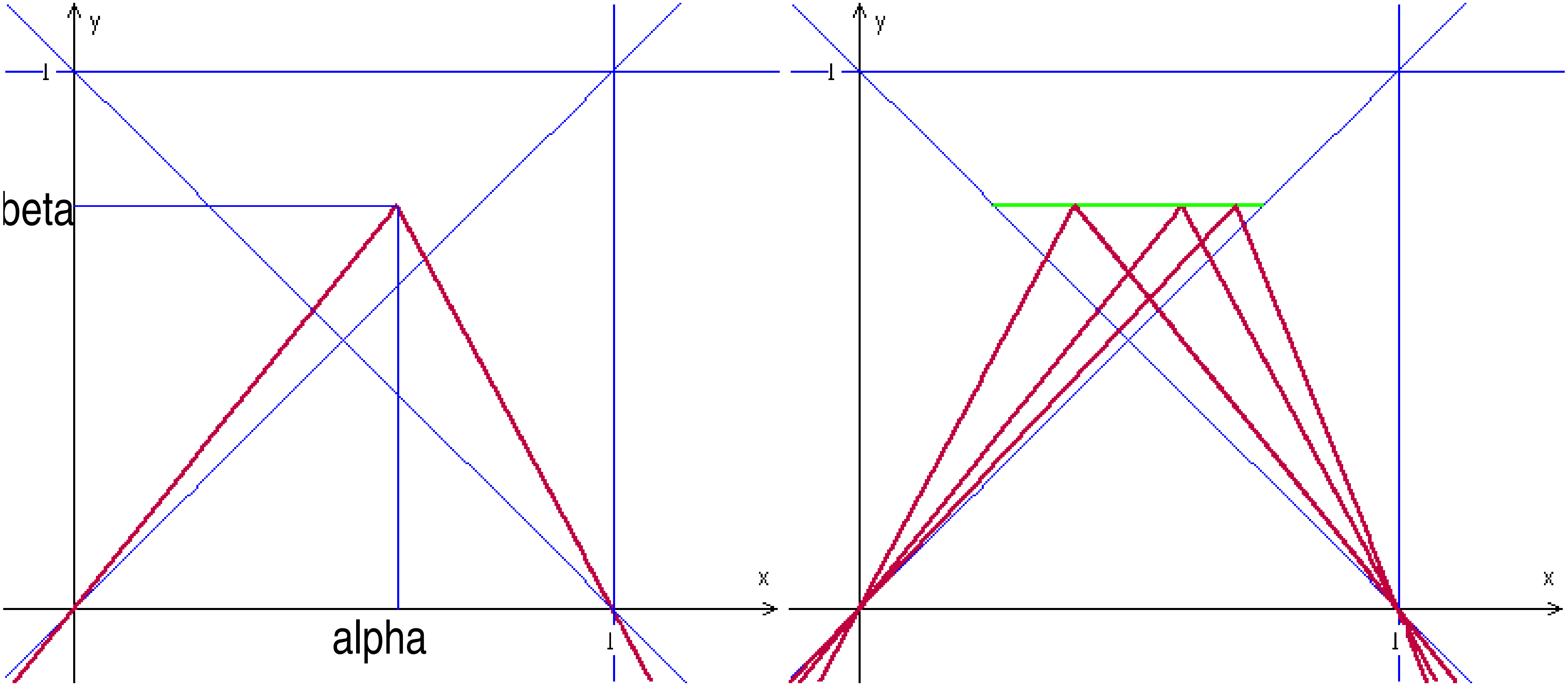
In fact, points $\lambda = \lambda(\alpha, \beta)$ and $\mu = \mu(\alpha, \beta)$ with fixed β satisfy $\frac{1}{\lambda} + \frac{1}{\mu} = \frac{1}{\beta}$, or $\lambda = \frac{1}{\frac{1}{\beta} - \frac{1}{\mu}}$ and hence λ is a monotone decreasing function of μ along curves in the (μ, λ) plane corresponding to horizontal line segments in the (α, β) plane.



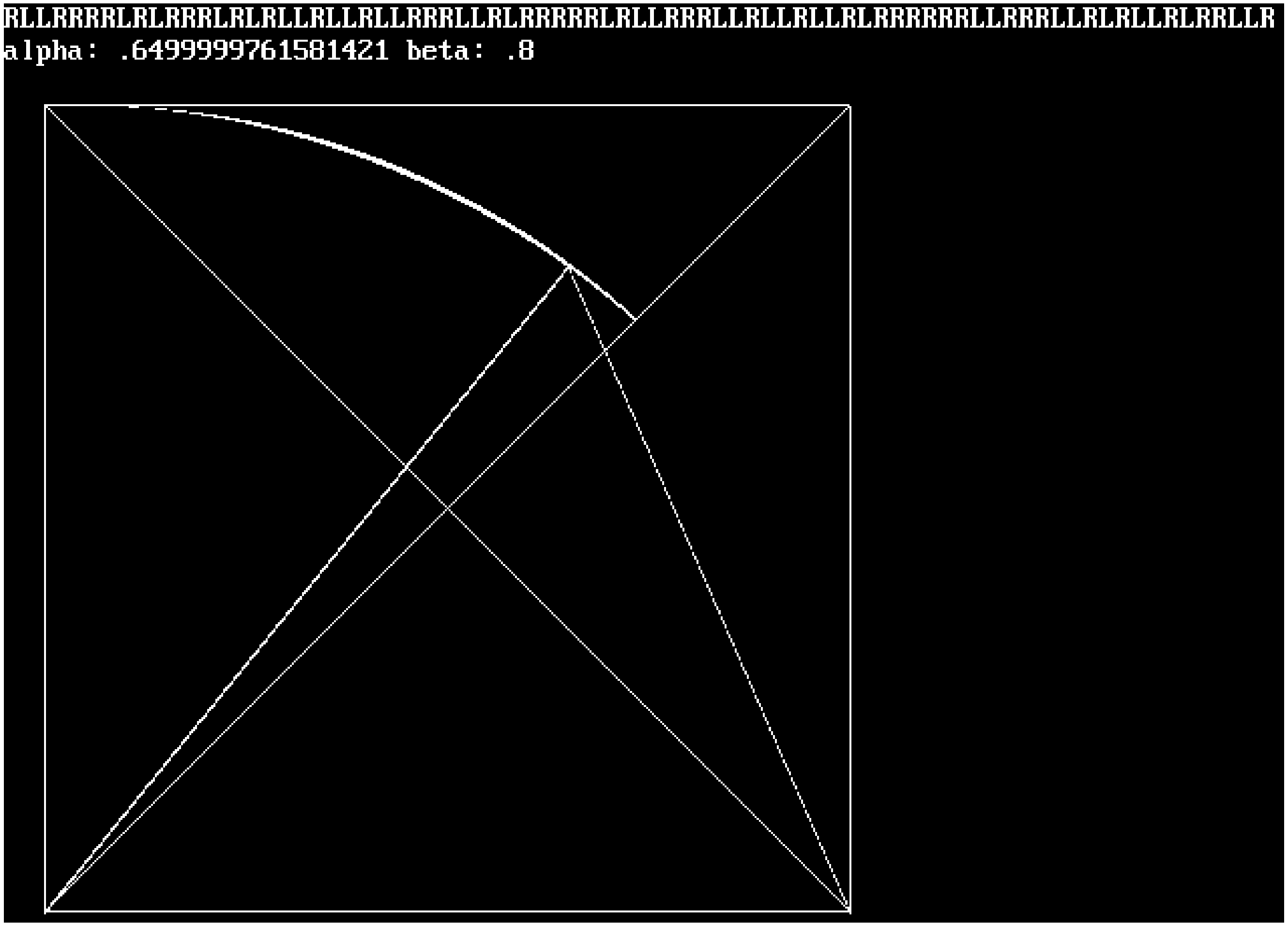


In fact, points $\lambda = \lambda(\alpha, \beta)$ and $\mu = \mu(\alpha, \beta)$ with fixed β satisfy $\frac{1}{\lambda} + \frac{1}{\mu} = \frac{1}{\beta}$, or $\lambda = \frac{1}{\frac{1}{\beta} - \frac{1}{\mu}}$ and hence λ is a monotone decreasing function of μ along curves in the (μ, λ) plane corresponding to horizontal line segments in the (α, β) plane.
If α is fixed and β increases then both $\lambda = \frac{\beta}{\alpha}$ and $\mu = \frac{\beta}{1-\alpha}$ increase. Therefore, the monotonicity result in [MV] implies that all the functions $g(\beta) = h(\alpha, \beta)$ are monotone increasing in our parameter range.
This parameter range $0.5 < \beta < 1,\ \alpha \in (1 - \beta, \beta)$ corresponds to the region bounded by the curves $\mu = 1,\ \lambda = 1$ and $\frac{1}{\mu} + \frac{1}{\lambda} = 1$ in the (μ, λ)-plane.

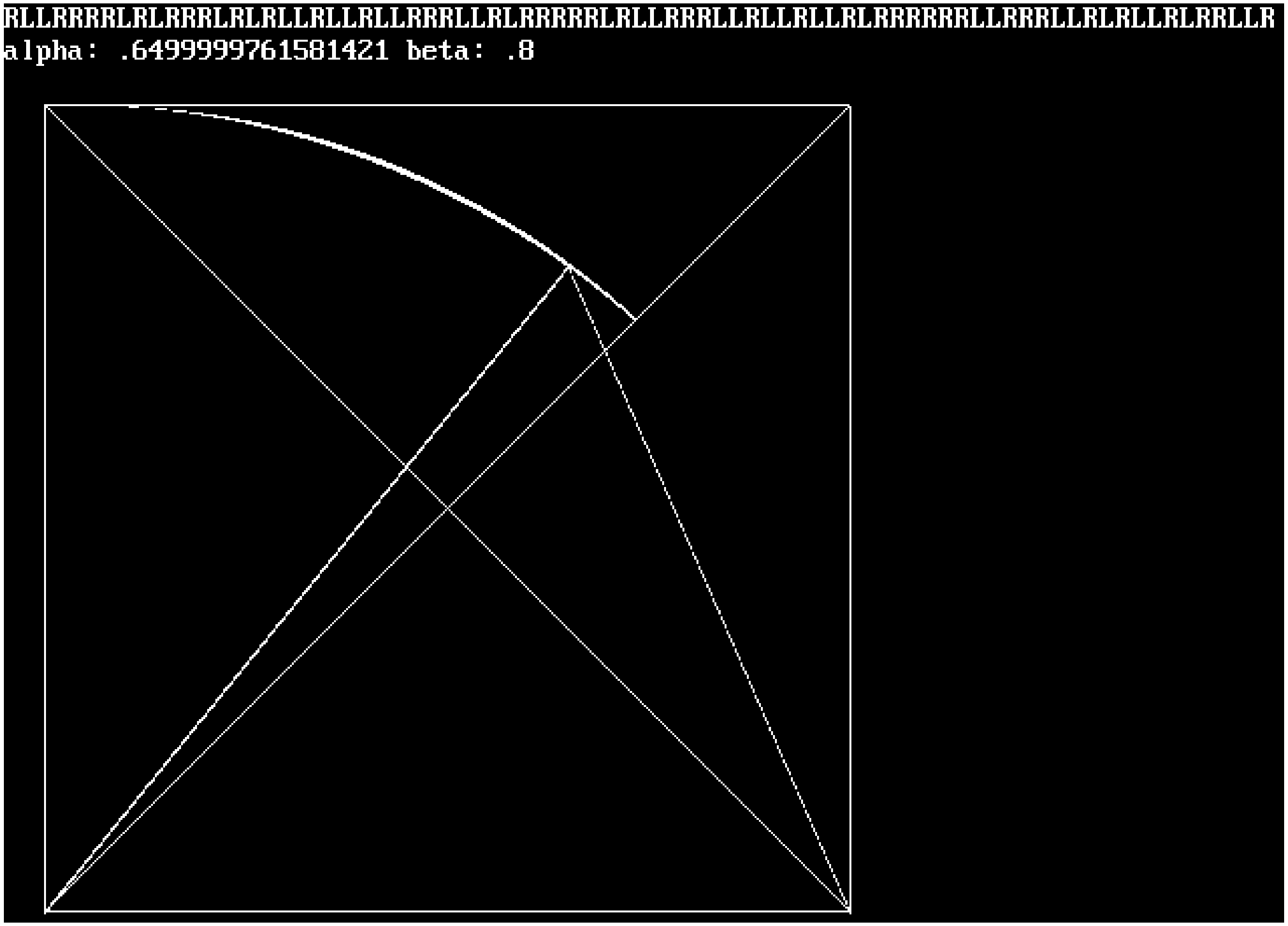


We prove:

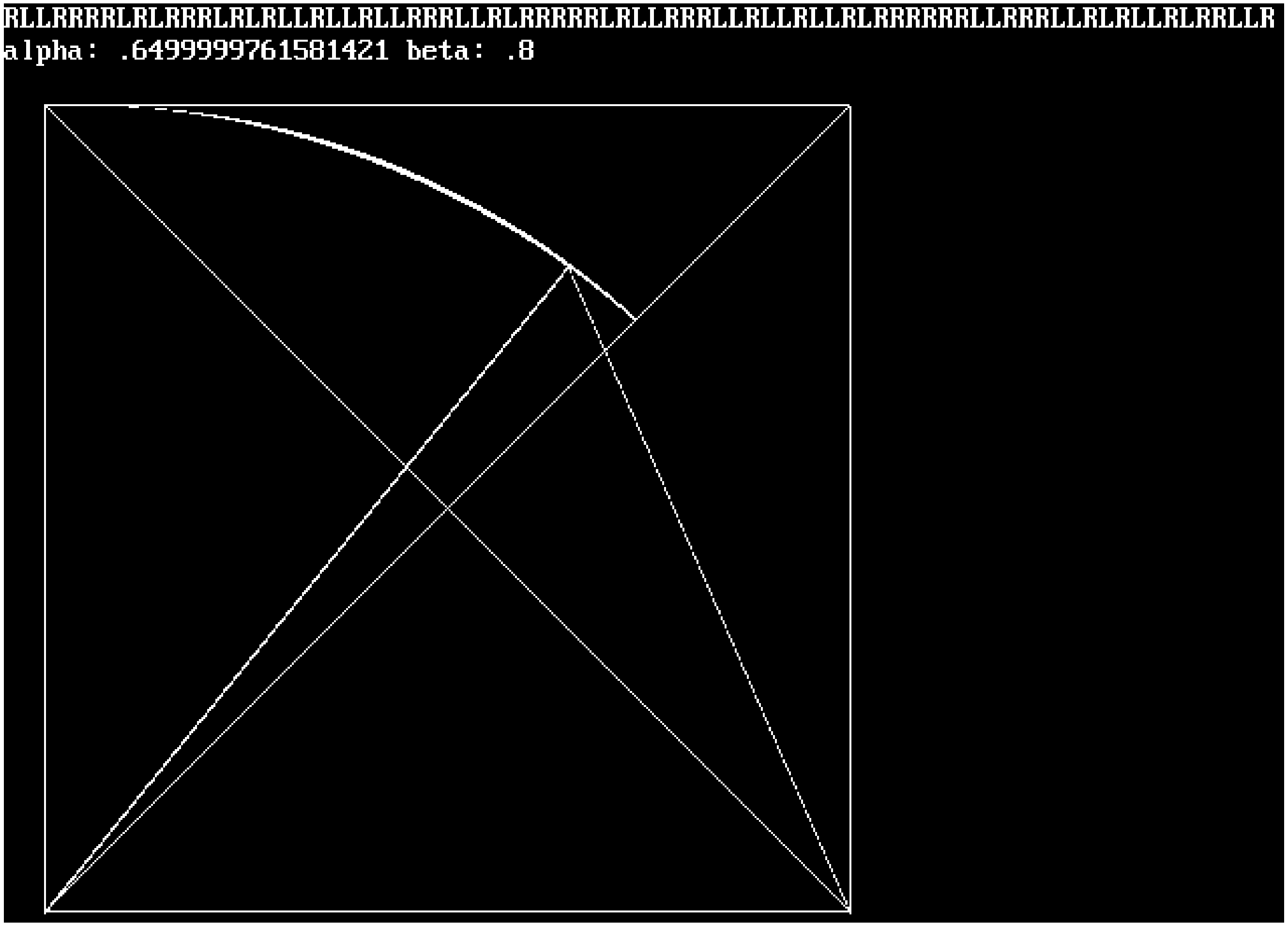
T.: *For any fixed* $\beta \in (0.5, 1)$ *the function* $h(\alpha) = h(\alpha, \beta)$ *is strictly monotone increasing on* $(1 - \beta, \beta)$.



We denote by $K(\alpha, \beta)$ the kneading sequence of $T(\alpha, \beta)$.



We denote by $K(\alpha, \beta)$ the kneading sequence of $T(\alpha, \beta)$.
If β is fixed we simply write $K(\alpha)$ and $T(\alpha)$.
If $K(\alpha) = A_1 A_2 \dots$ then $K_n(\alpha) = A_n \in \{R, L, C\}$.



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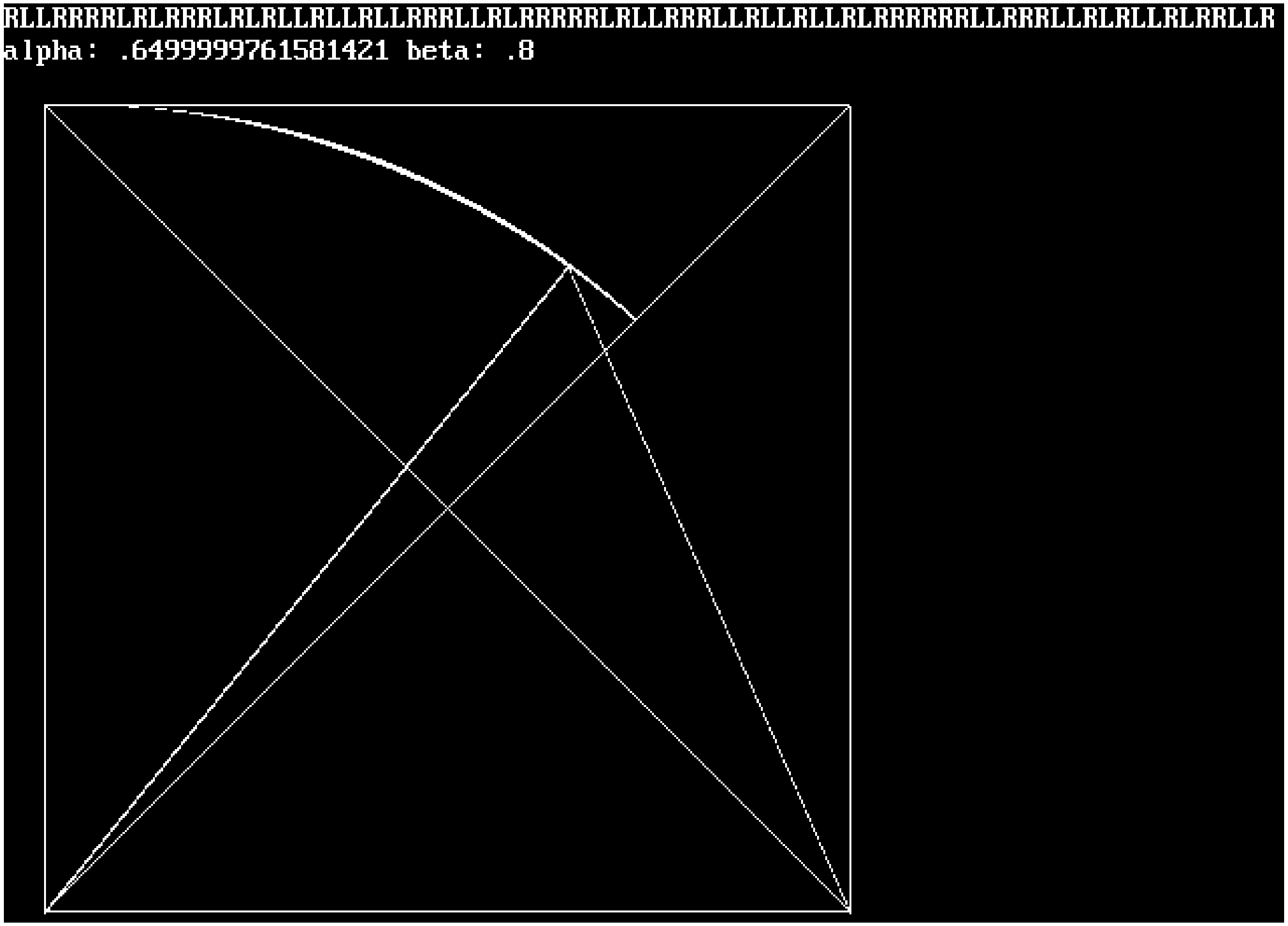
We denote by $\mathfrak{M}$ the class of kneading sequences $K(0.5,\beta)$, $\beta \in (0.5, 1]$, this corresponds to the kneading sequences of functions $F_{\mu,\mu}$ with $1 < \mu \leq 2$.



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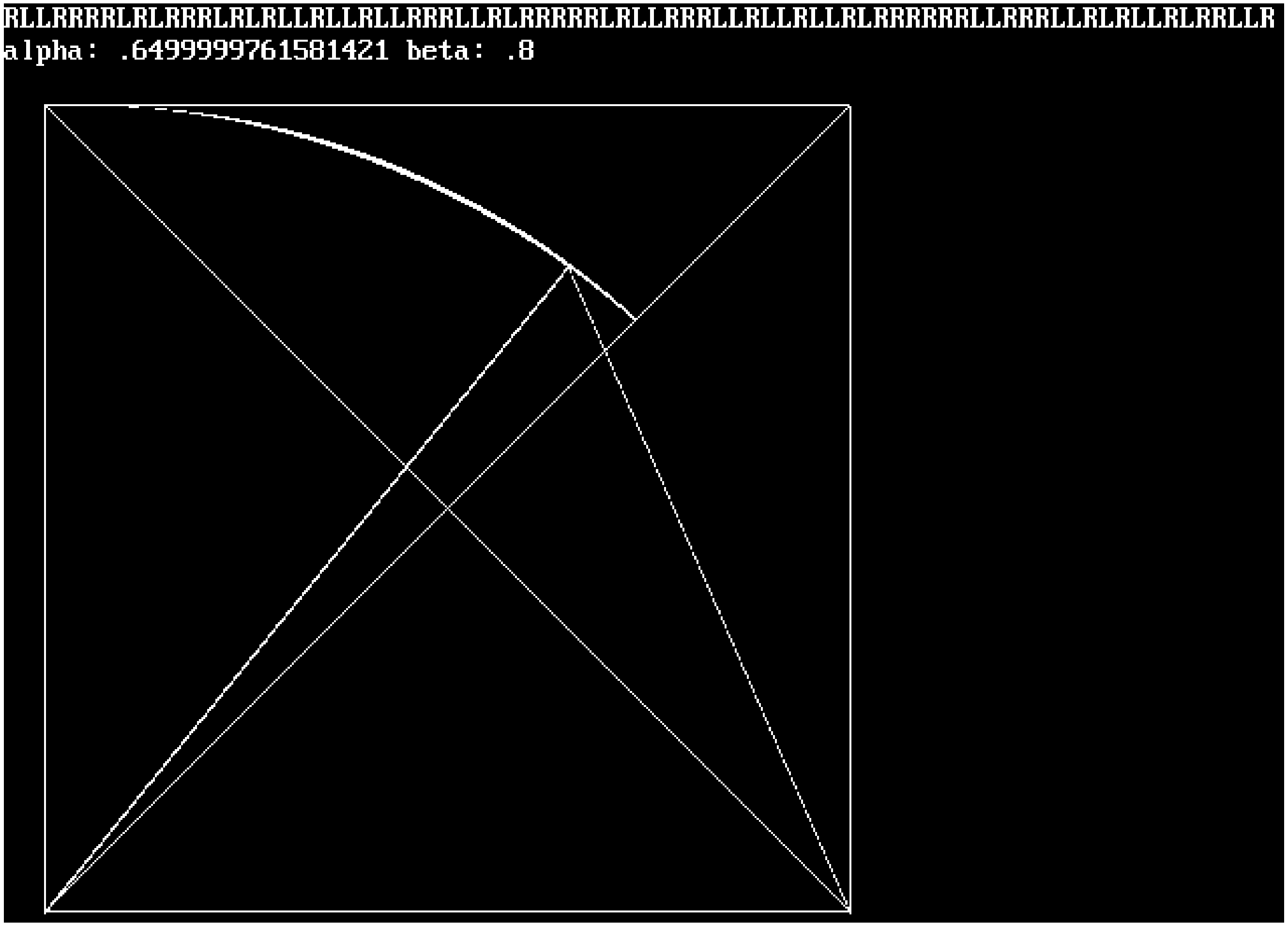
Results of M–V imply: **T**.: For each $\underline{M} \in \mathfrak{M}$ there exist two numbers $\alpha_1(\underline{M}) < \alpha_2(\underline{M})$ and a continuous function $\Psi_{\underline{M}} : (\alpha_1(\underline{M}), \alpha_2(\underline{M})) \to U$ such that for $(\alpha, \beta) \in U$ we have $K(\alpha, \beta) = \underline{M}$ if and only if $\beta = \Psi_{\underline{M}}(\alpha)$.

The graphs of the functions $\Psi_{\underline{M}}$ fill up the whole set U. Moreover, $\lim_{\alpha \to \alpha_1(\underline{M})+} \Psi_{\underline{M}}(\alpha) = 1$ if $\underline{M} \succcurlyeq RLR^{\infty}$.   If $\underline{M} \prec RLR^{\infty}$ then the curve $(\alpha, \Psi_{\underline{M}}(\alpha))$ converges to a point on the line segment $\{(\alpha, 1-\alpha) : 0 < \alpha < \frac{1}{2}\}$.



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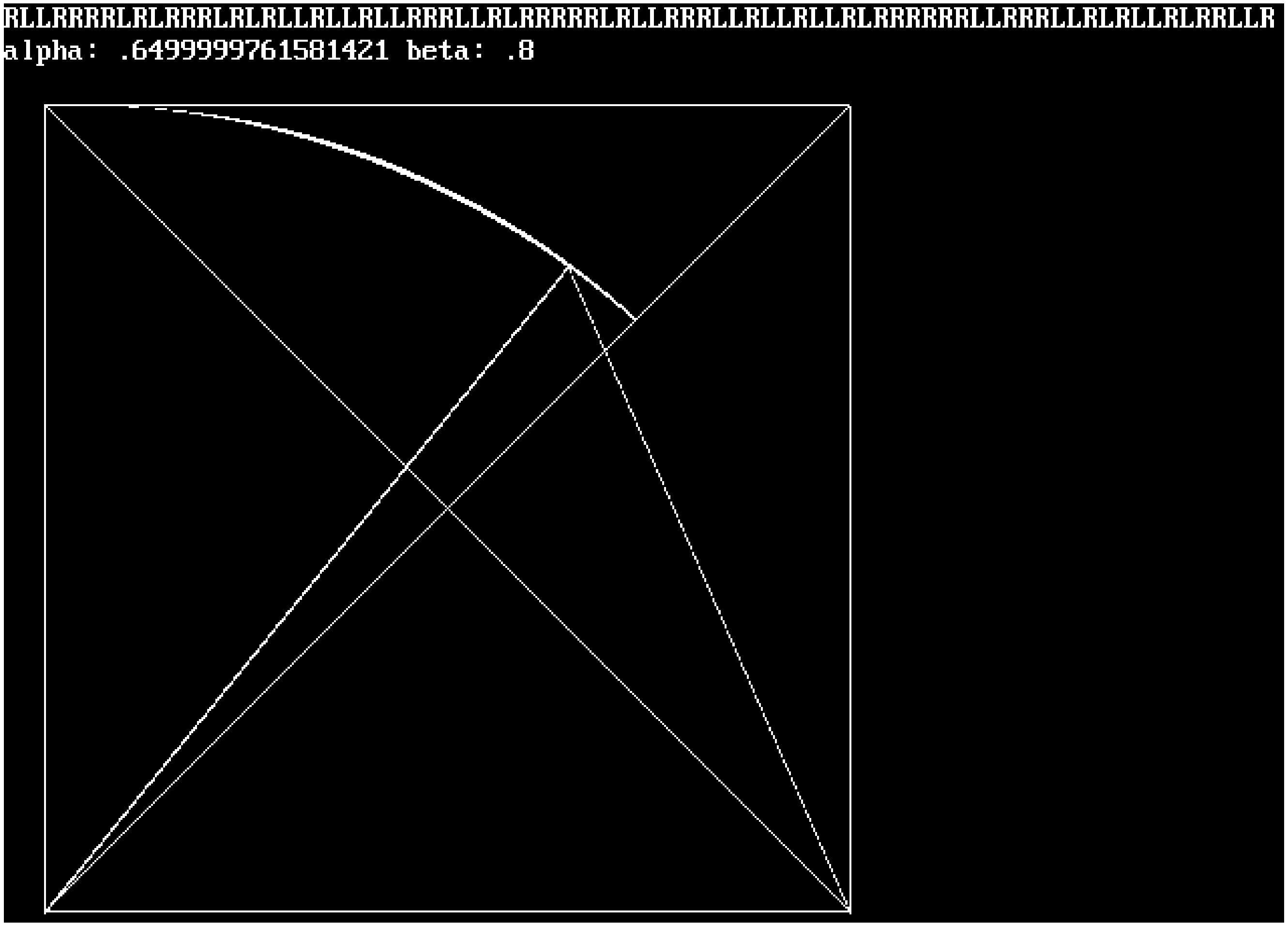

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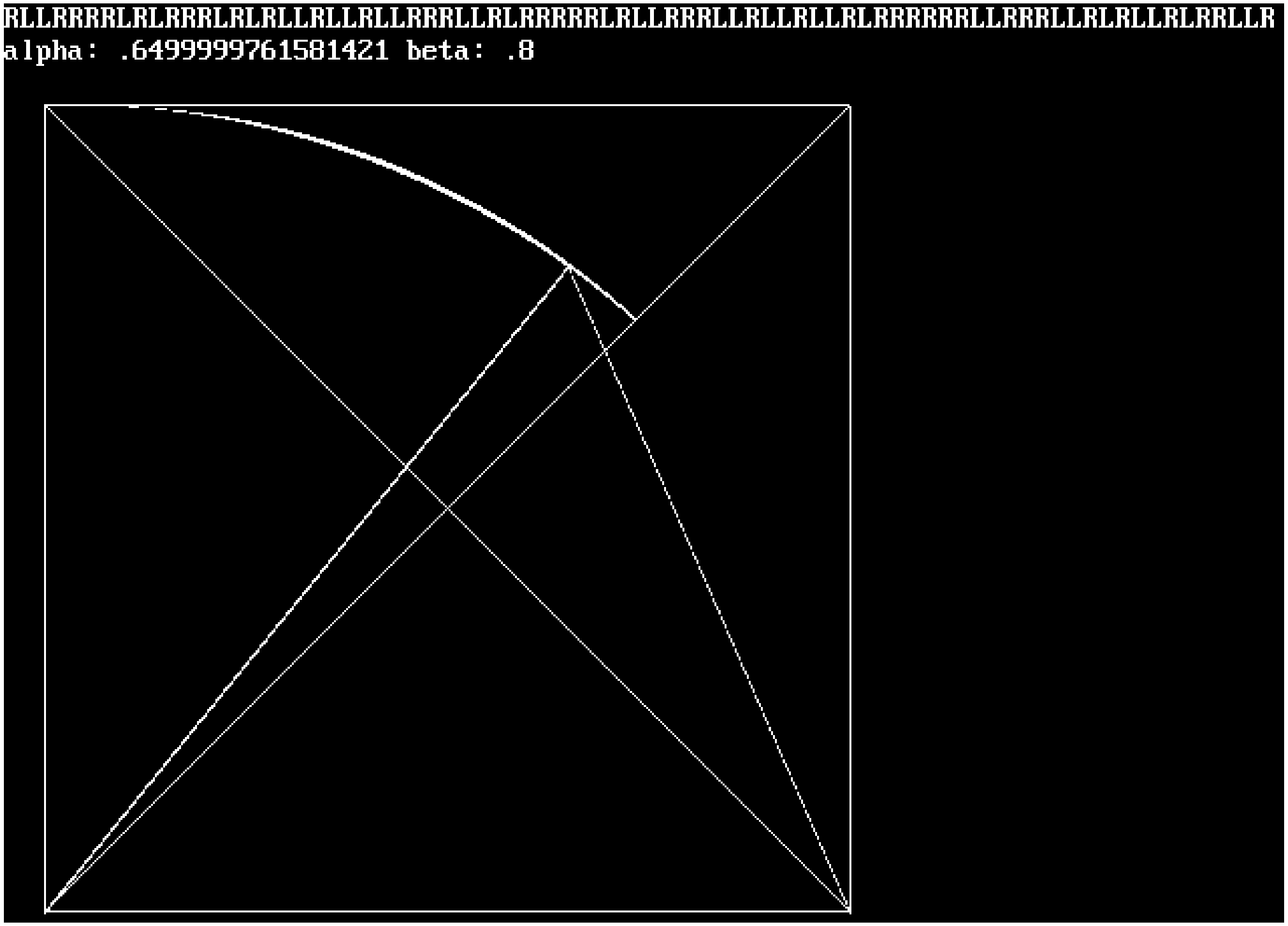
We denote by $\mathfrak{M}_{R\infty}$ the set of those $\underline{M} \in \mathfrak{M}$ which are of the form $\underline{M} = A_1 A_2 \ldots A_{n-1} R^\infty$.

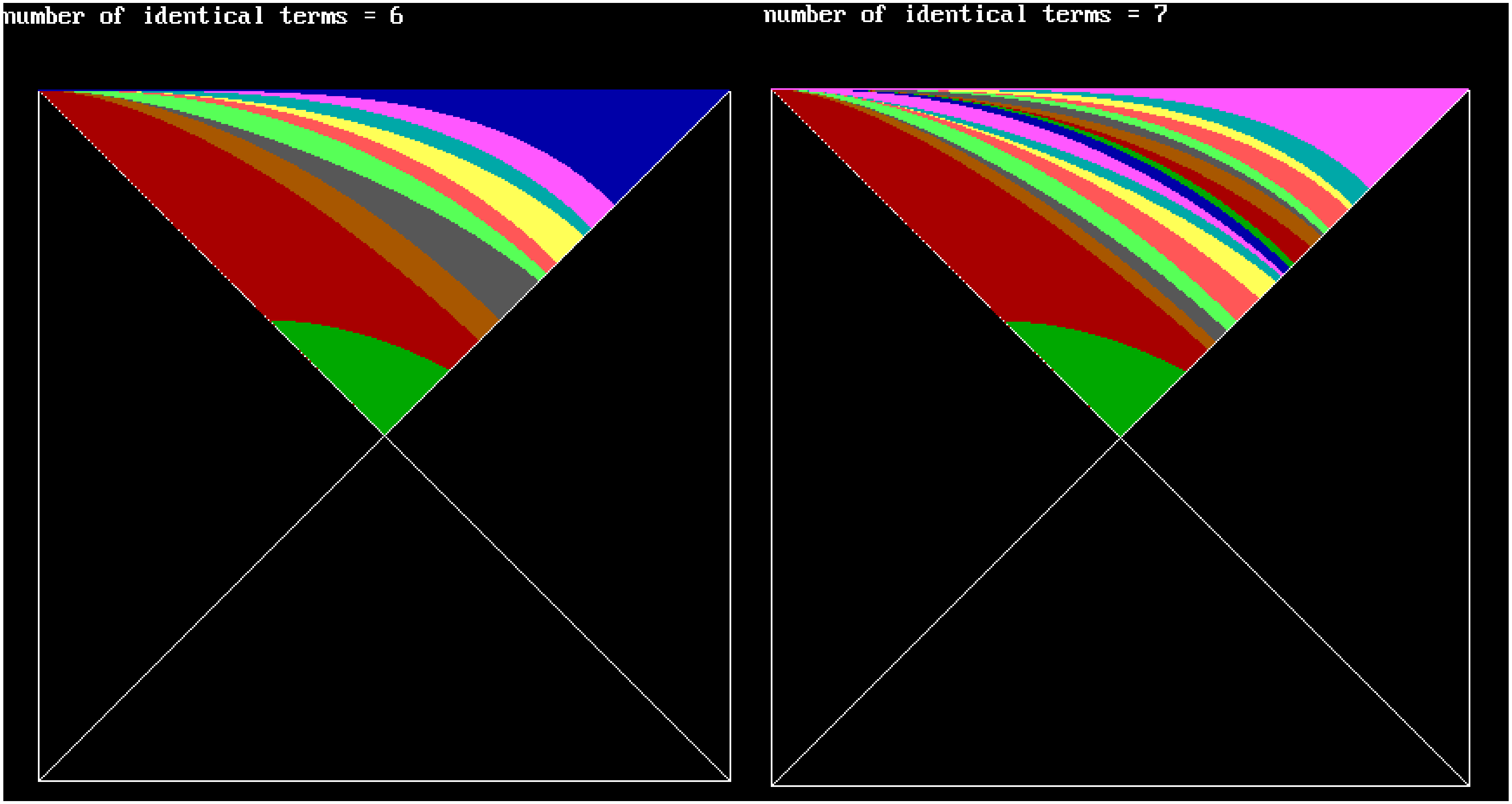


We prove:

If $\underline{M} \in \mathfrak{M} \setminus \{RL^\infty\}$ then $\Psi_{\underline{M}}(\alpha)$ is strictly monotone decreasing.

This implies: The function $h(\alpha) = h(\alpha, \beta)$ is monotone increasing on $(1-\beta, \beta)$.

We also show that close to $(1,1)$ the curves $\Psi_{\underline{M}}(\alpha)$ are almost perpendicular to the $y = x$ line.



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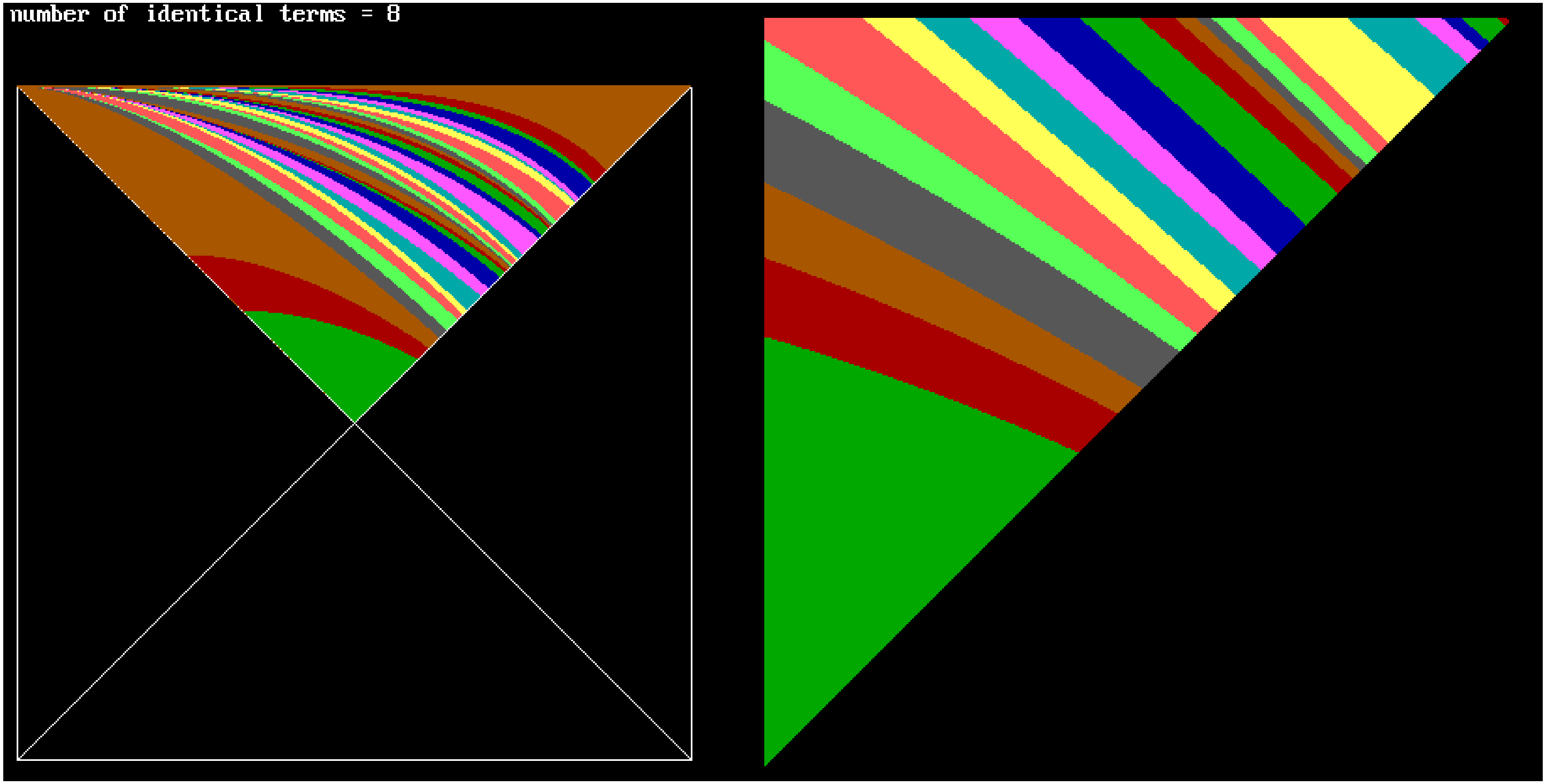

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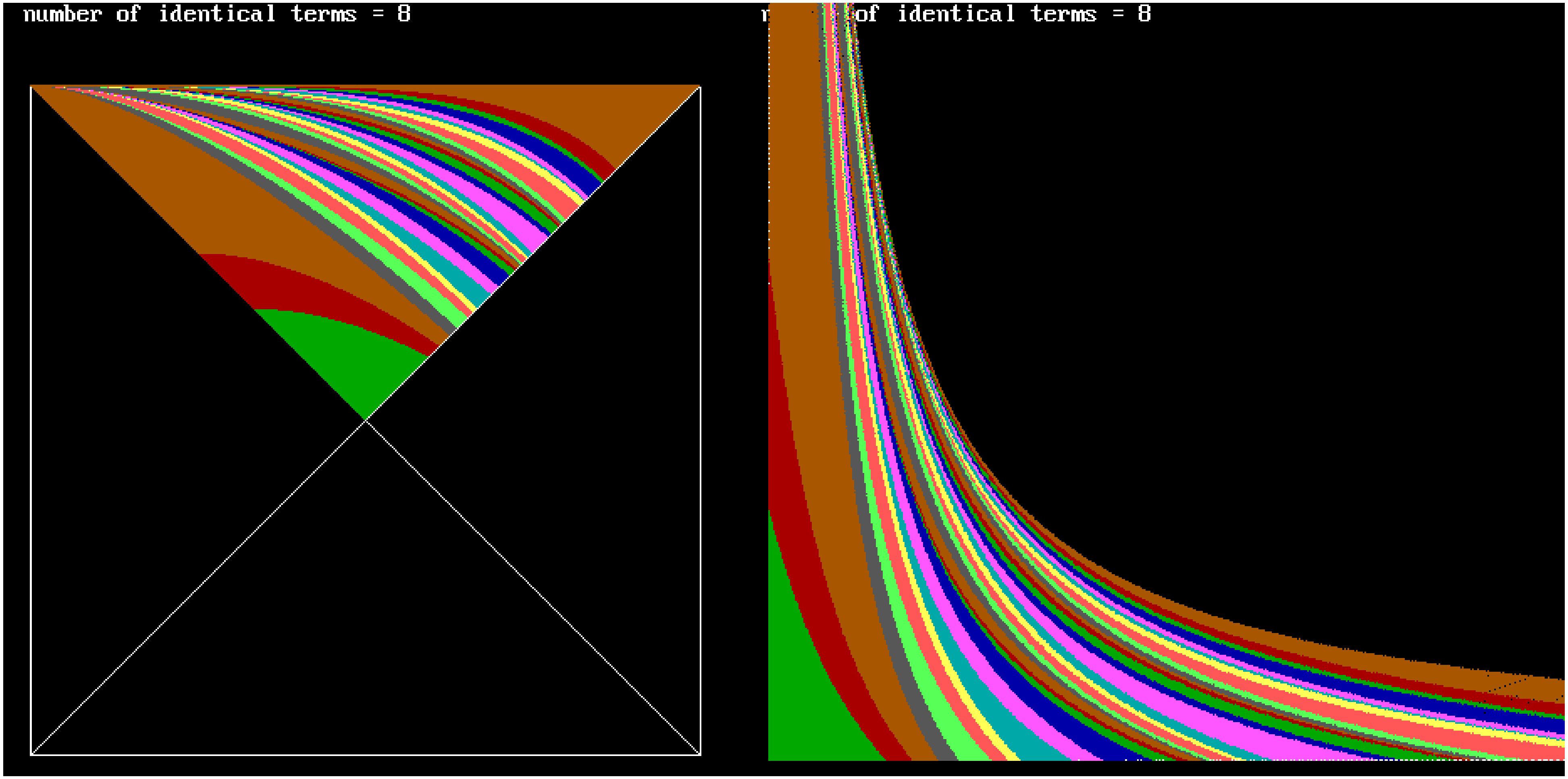
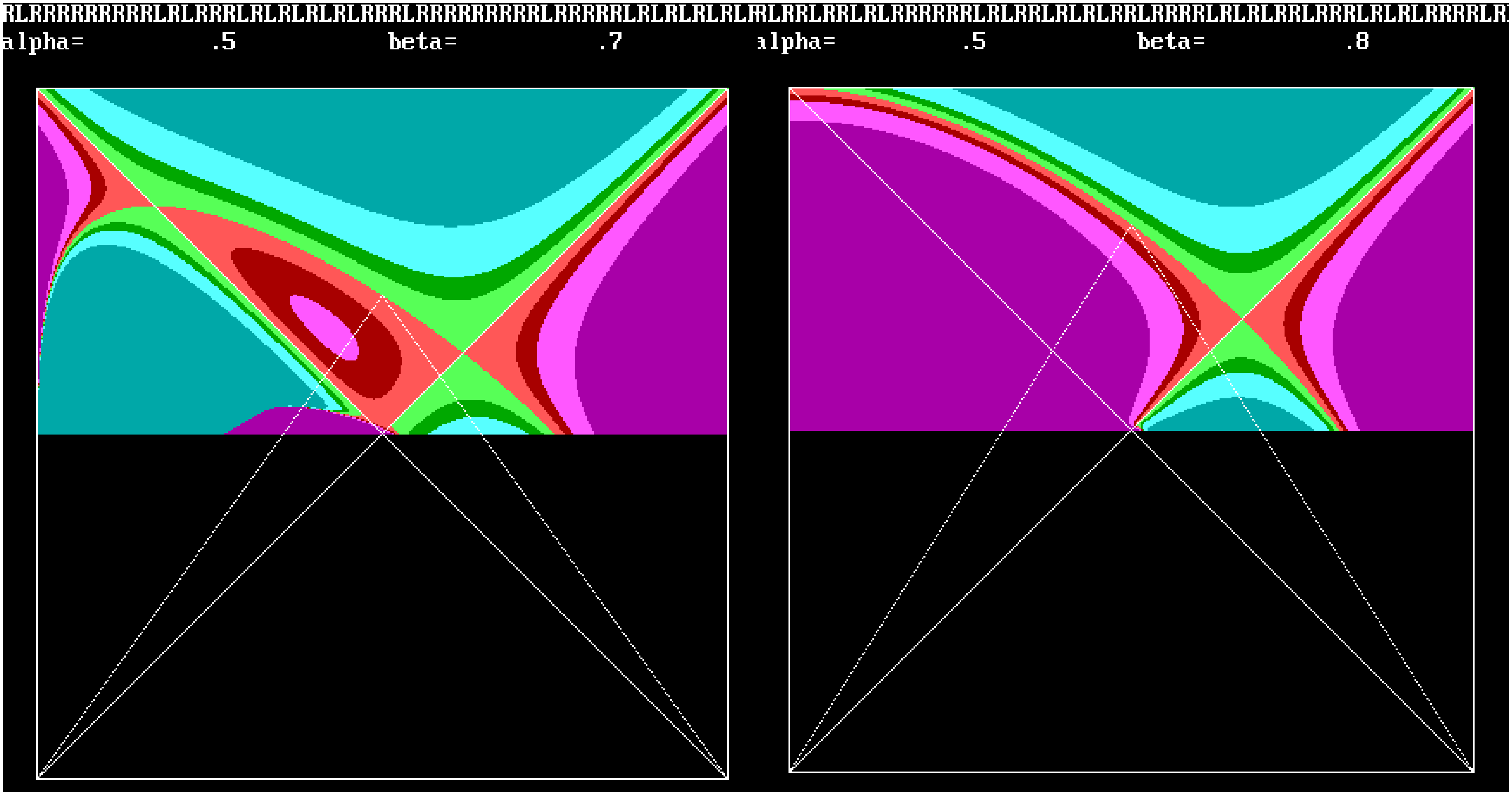
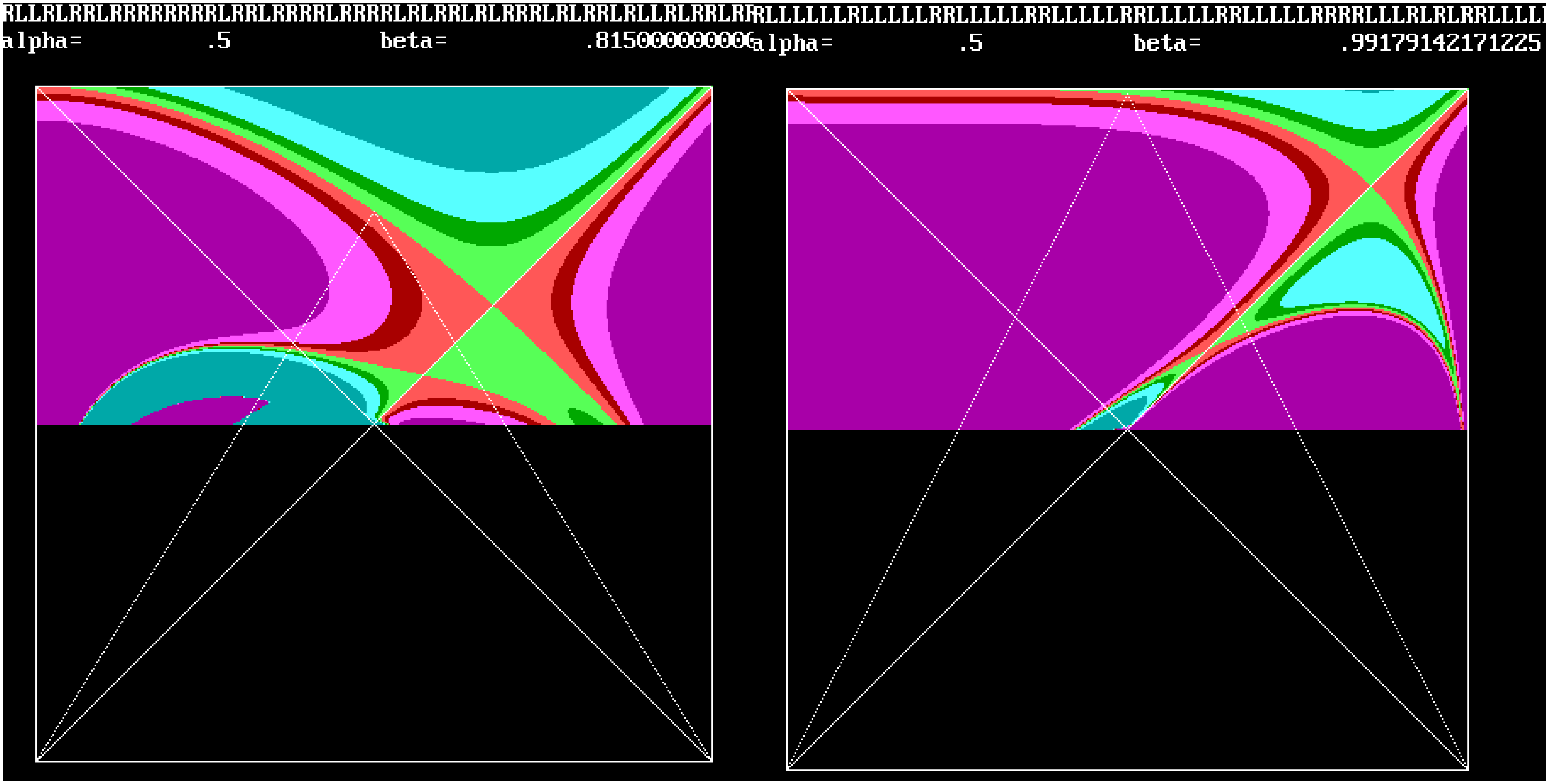


Equi-kneading regions by using the „square-parametrization" and by the parametrization used in the Misiurewicz–Visinescu paper.



L.: *Suppose* $\underline{M} \in \mathfrak{M} \backslash \{RL^\infty\}$ *is given. Then there exists a function* $\Theta_{\underline{M}}(\alpha, \beta)$, $\Theta_{\underline{M}} : U \to \mathbb{R}$, *such that for* $(\alpha, \beta) \in U$ *from* $K(\alpha, \beta) = \underline{M}$ *it follows that* $\Theta_{\underline{M}}(\alpha, \beta) = 0$. *(Limitations on reverse implication!)*



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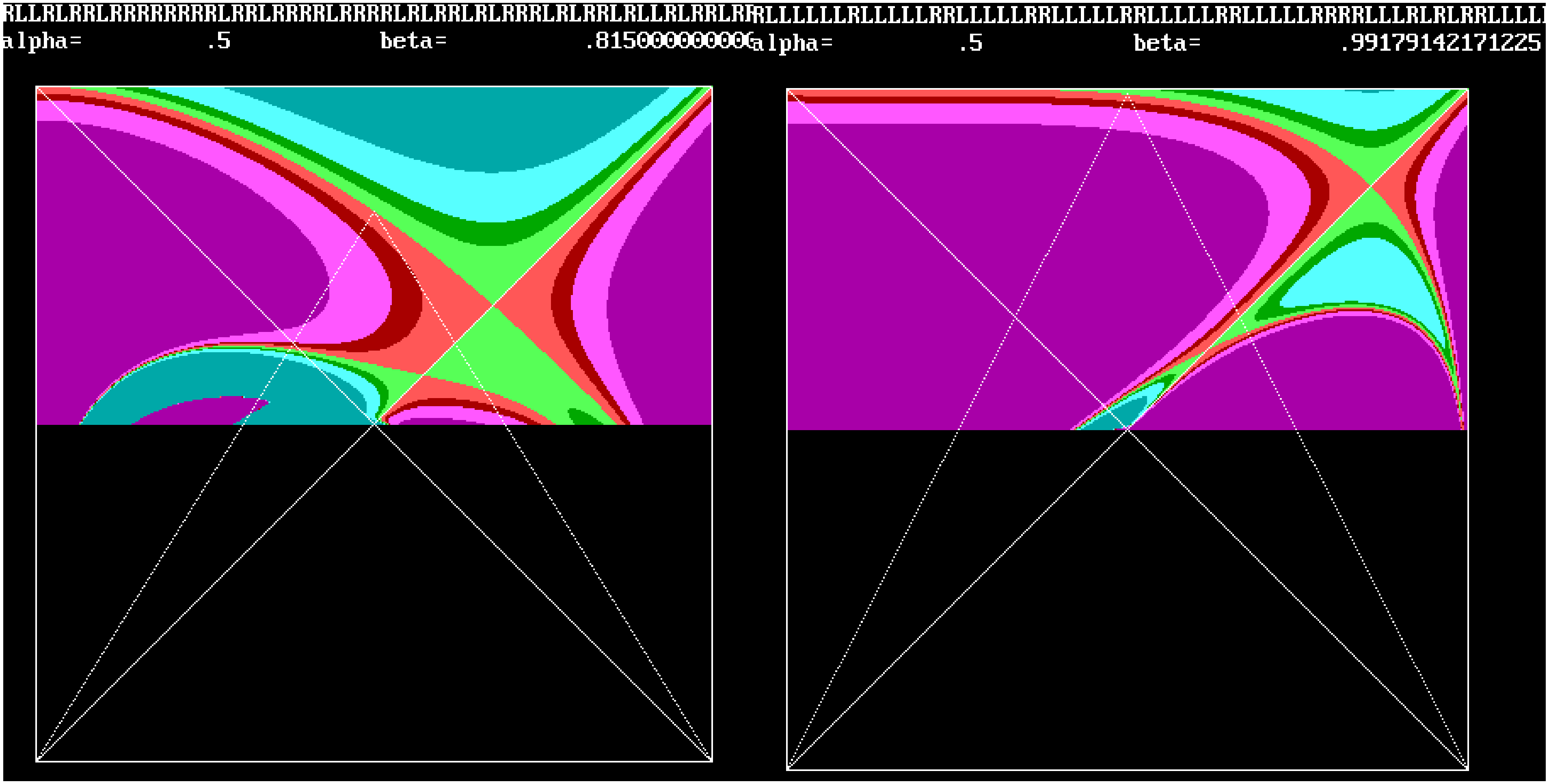
Moreover,
$$\Theta_{\underline{M}}(\alpha, \beta) = 1 - \beta + \sum_{k=1}^{\infty} \left(\frac{\alpha - 1}{\beta} \right)^k \left(\frac{\alpha}{\beta} \right)^{m_k}$$
where $m_1 > 0$, $m_k \le m_{k+1} \le m_k + m_1$, $k = 0, 1, \ldots$.



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If $K(\alpha,\beta) \in \mathfrak{M}_{R,\infty}$ then there exists n s. t. $m_{k+1} = m_k$ for $k \ge n$.

If $\underline{M}$ is infinite let $\underline{M}' = \underline{M}$ if $\underline{M}$ is finite, $\underline{M} = A_1 \dots A_{n-1} C$ then let $\underline{M}' = A_1 \dots A_{n-1} L A_1 \dots A_{n-1} L \dots$.

Recall that $A_1 = R$ and $A_2 = L$ for all $\underline{M} = K(\alpha, \beta), (\alpha, \beta) \in U$.

If $\underline{M} \notin \mathfrak{M}_{R,\infty}$ than there are infinite sequences (λ_j), (h_j) s. t. in $\underline{M}'$ the first R is followed by λ_1 many L's, then there are h_1 many R's then λ_2 many L's, then h_2 many R's etc., that is,

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$\Theta_{\underline{M}}(\alpha, \beta)$ can be written as

$$\Theta_{\underline{M}}(\alpha, \beta) = (1 - \beta) + \left(\frac{\alpha - 1}{\beta}\right)\left(\frac{\alpha}{\beta}\right)^{\lambda_1} + \dots + \left(\frac{\alpha - 1}{\beta}\right)^{h_1}\left(\frac{\alpha}{\beta}\right)^{\lambda_1} +$$

$$\dots + \left(\frac{\alpha - 1}{\beta}\right)^{h_1 + \dots + h_{n-1} + 1}\left(\frac{\alpha}{\beta}\right)^{\lambda_1 + \dots + \lambda_n} + \dots + \left(\frac{\alpha - 1}{\beta}\right)^{h_1 + \dots + h_n}\left(\frac{\alpha}{\beta}\right)^{\lambda_1 + \dots + \lambda_n} + \dots .$$

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$$\Theta_{\underline{M}}(\alpha,\beta) = 1 - \beta + \sum_{k=1}^{\infty} \left(\frac{\alpha - 1}{\beta}\right)^{k} \left(\frac{\alpha}{\beta}\right)^{m_k}$$

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We need this to verify:

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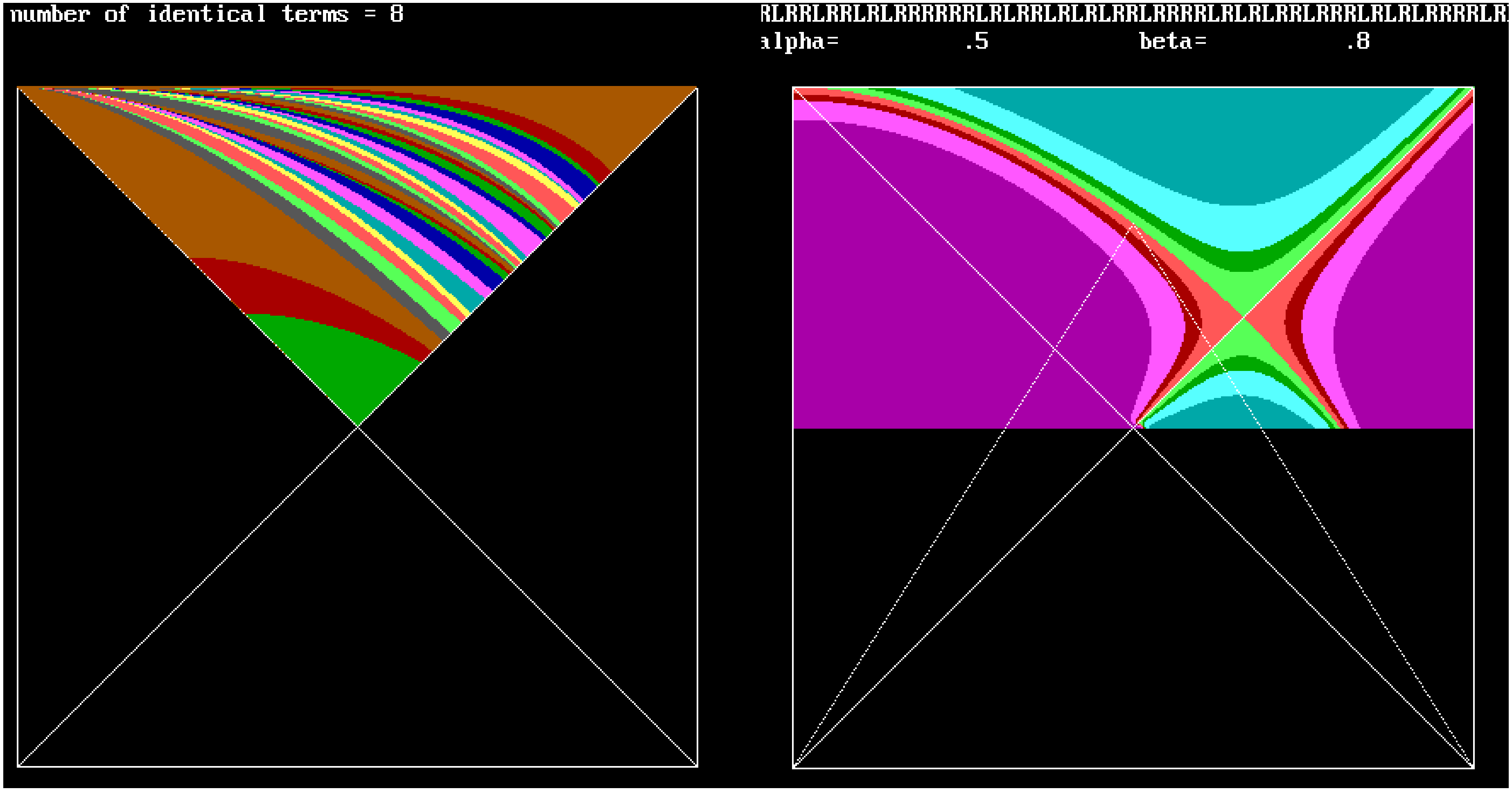
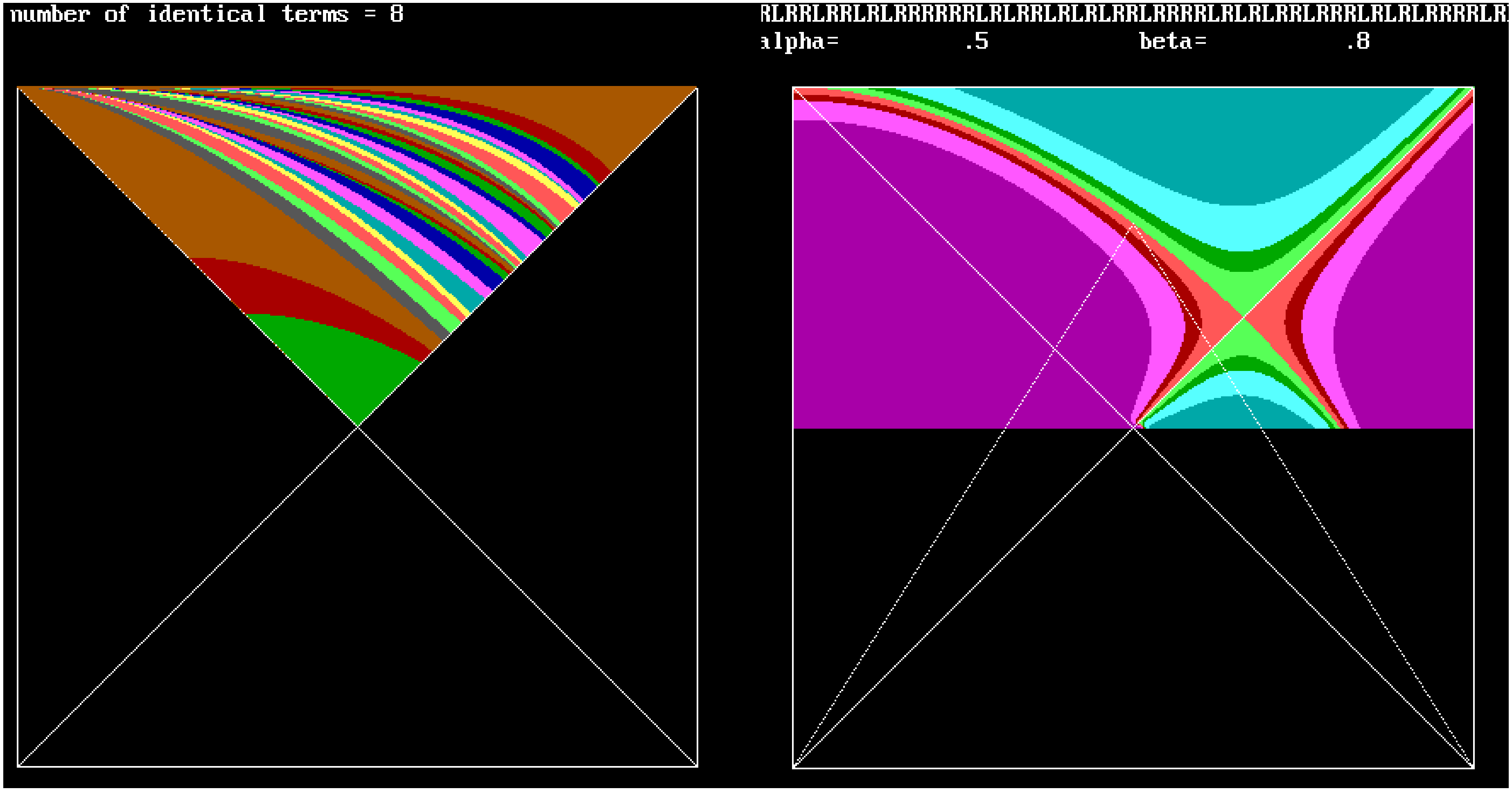


We prove that for $\underline{M} \in \mathfrak{M} \setminus \{RL^\infty\}$ the curves $(\alpha, \Psi_{\underline{M}}(\alpha))$, $\alpha \in (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$ are strictly monotone decreasing.



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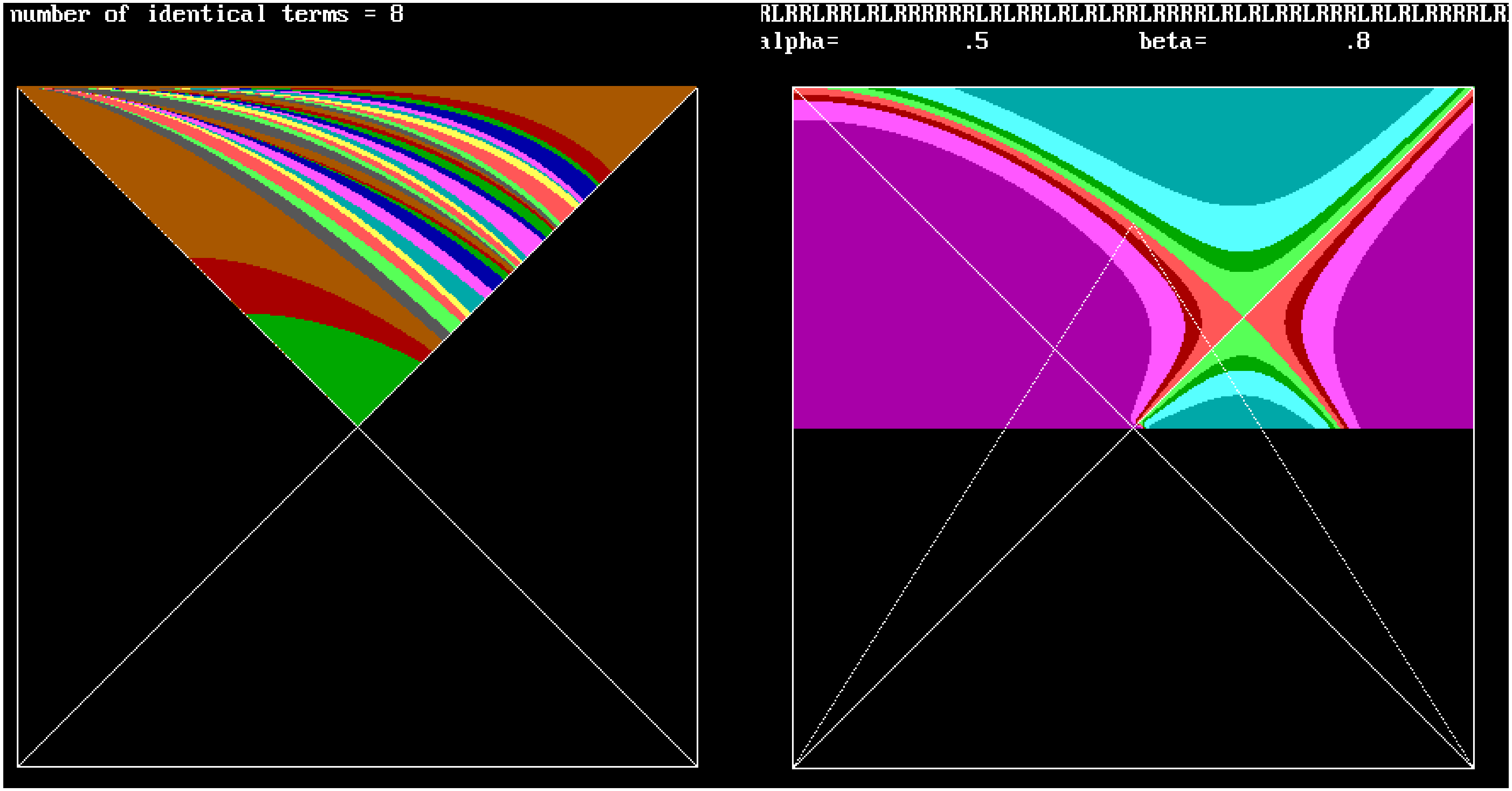
Since $\frac{1}{2} < \Psi_{\underline{M}}\left(\frac{1}{2}\right) < 1$ these curves reach the boundary of U at a point on the line segment $\{(\beta,\beta) : \frac{1}{2} < \beta < 1\}$, in fact $\beta_{\underline{M}} = \alpha_2(\underline{M})$ and $\lim_{\alpha\to\alpha_2(\underline{M})} \Psi_{\underline{M}}(\alpha) = \beta_{\underline{M}}$.



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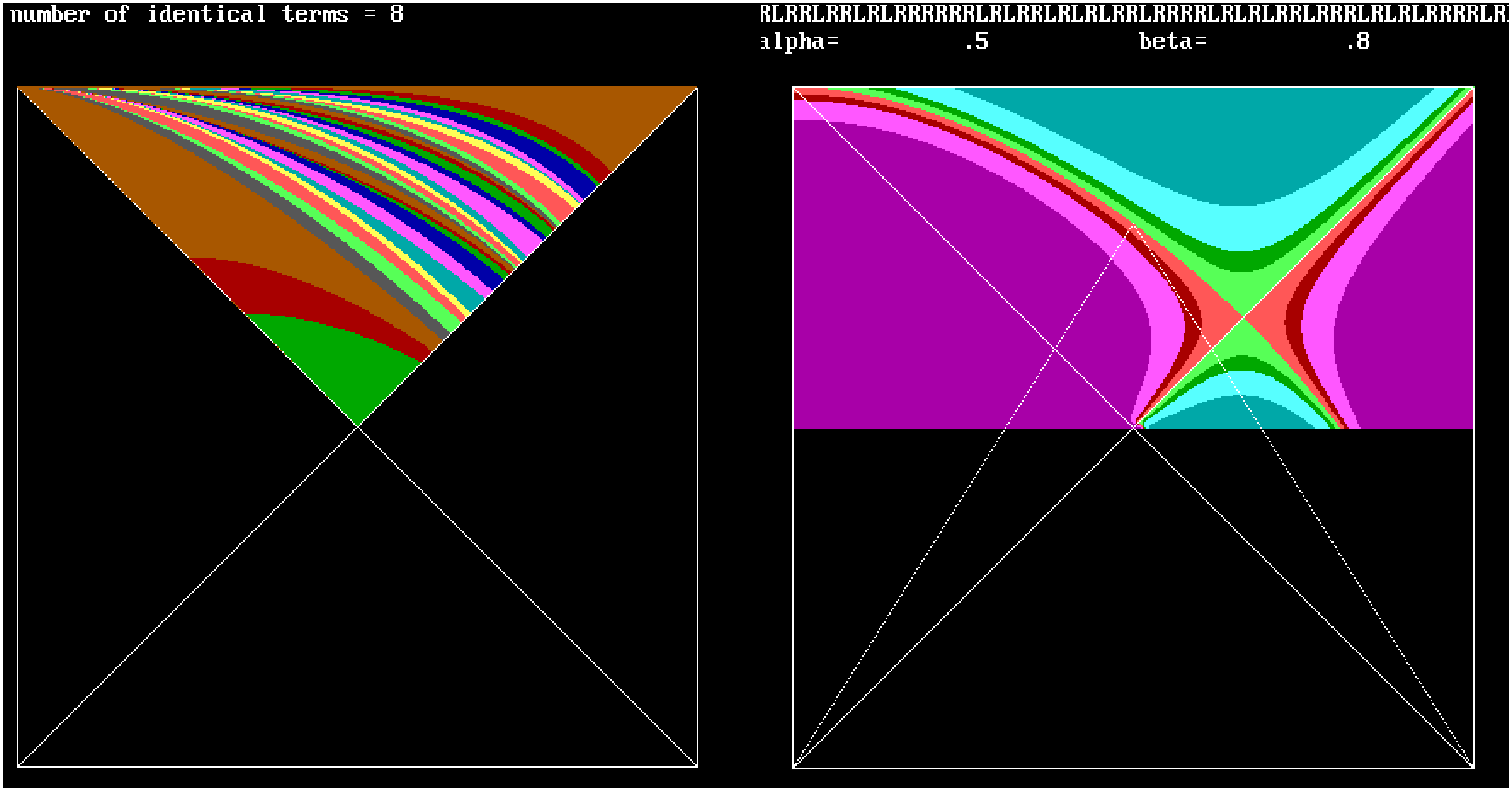

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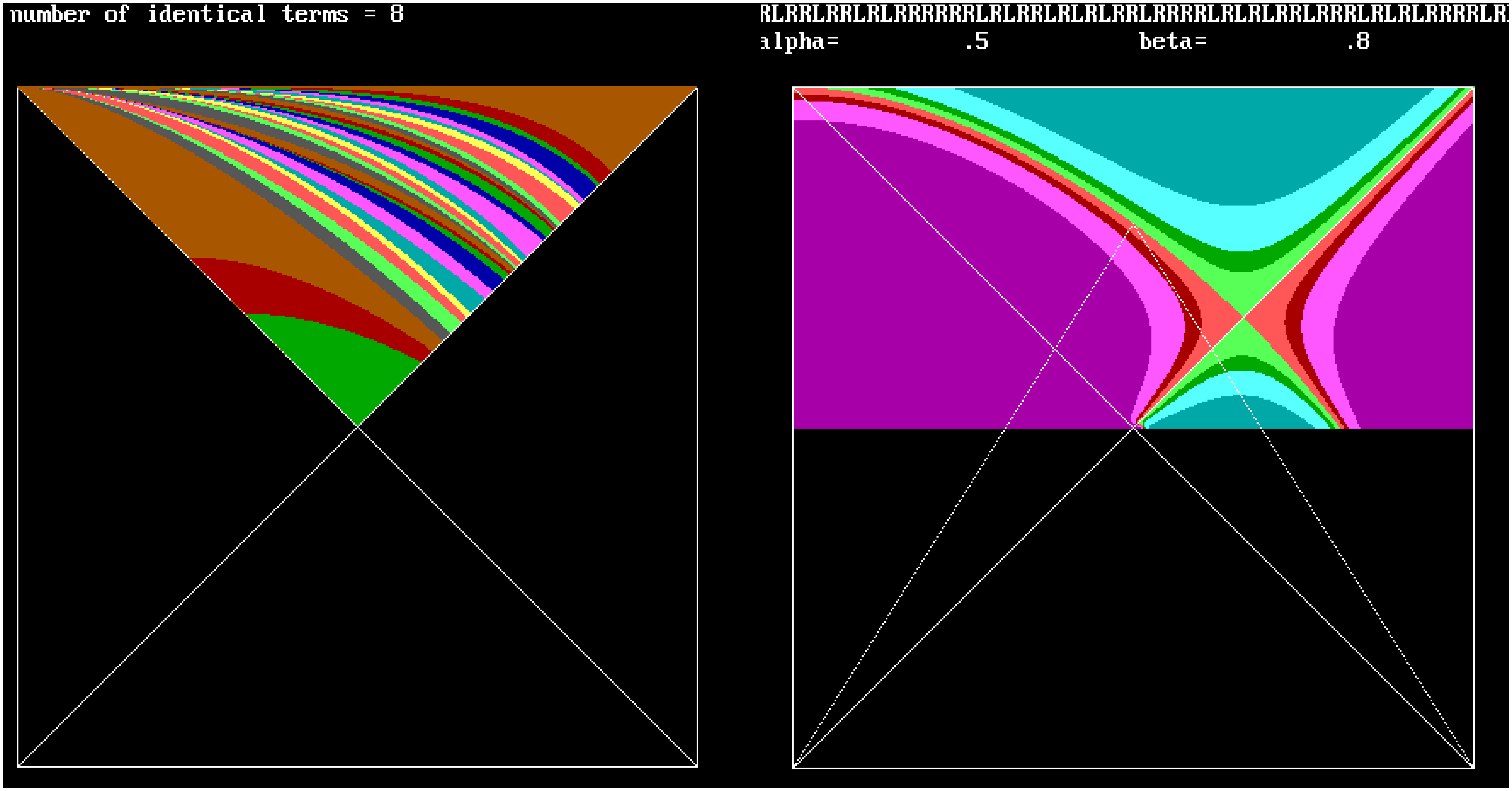

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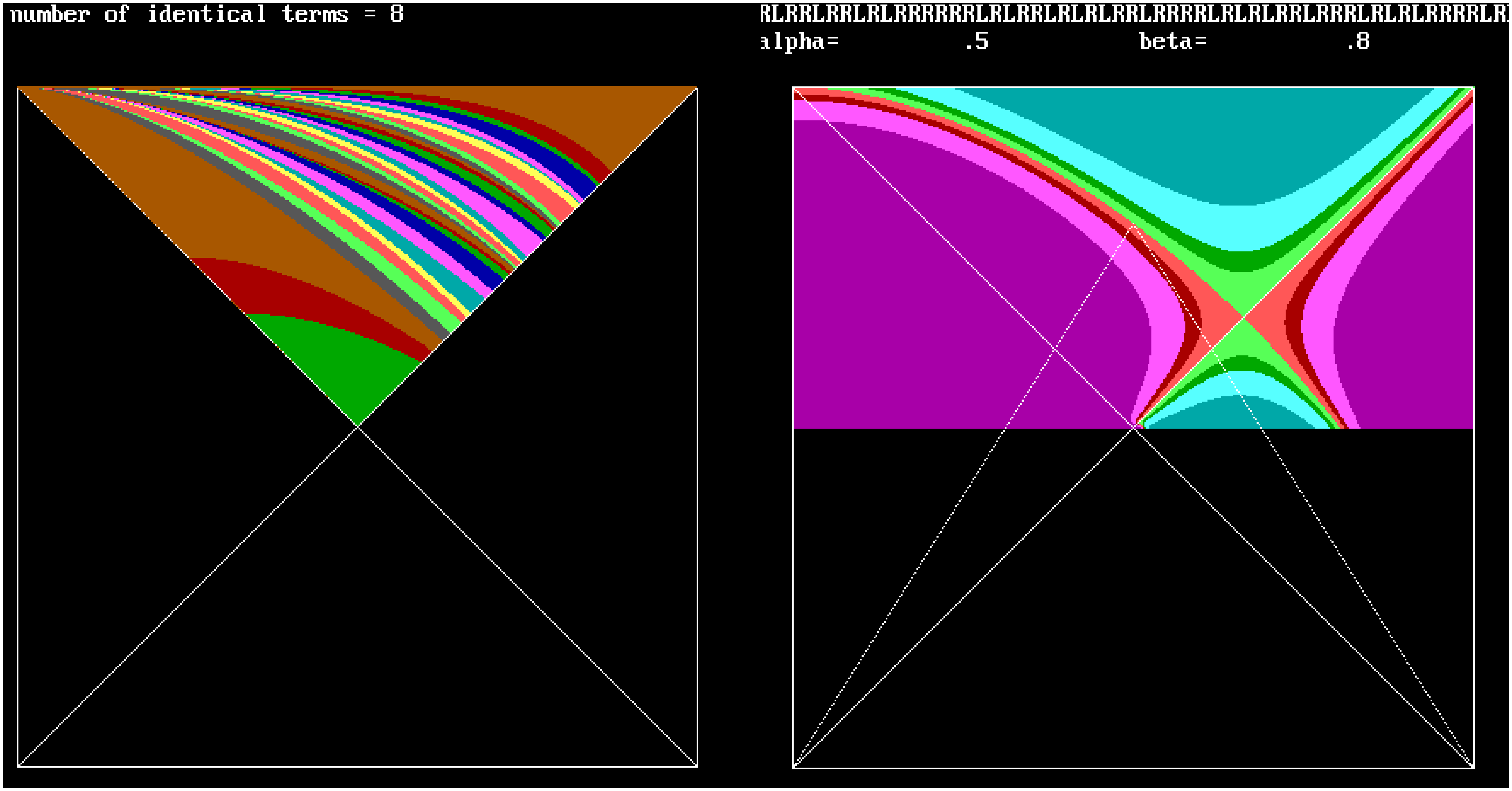

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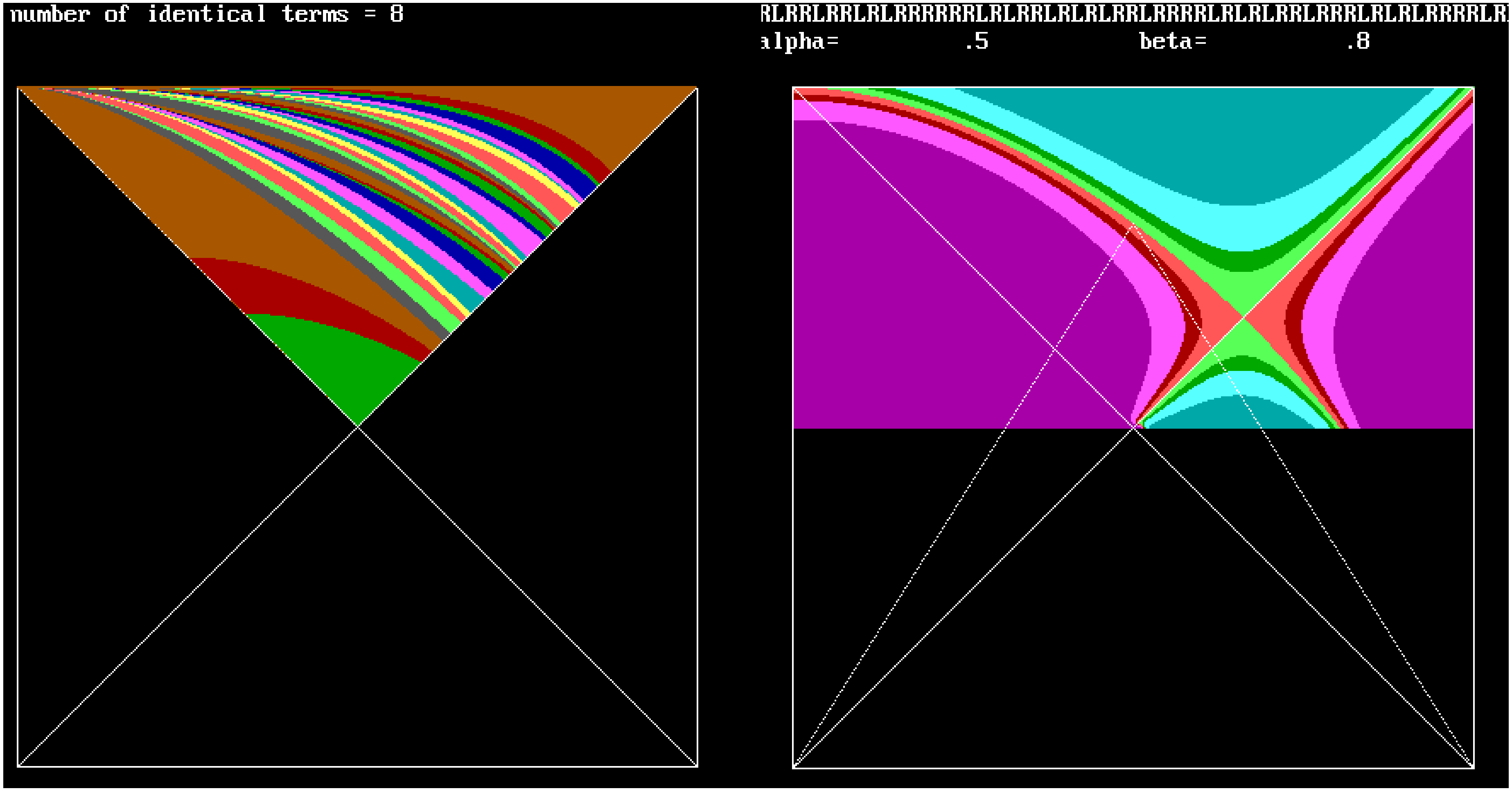

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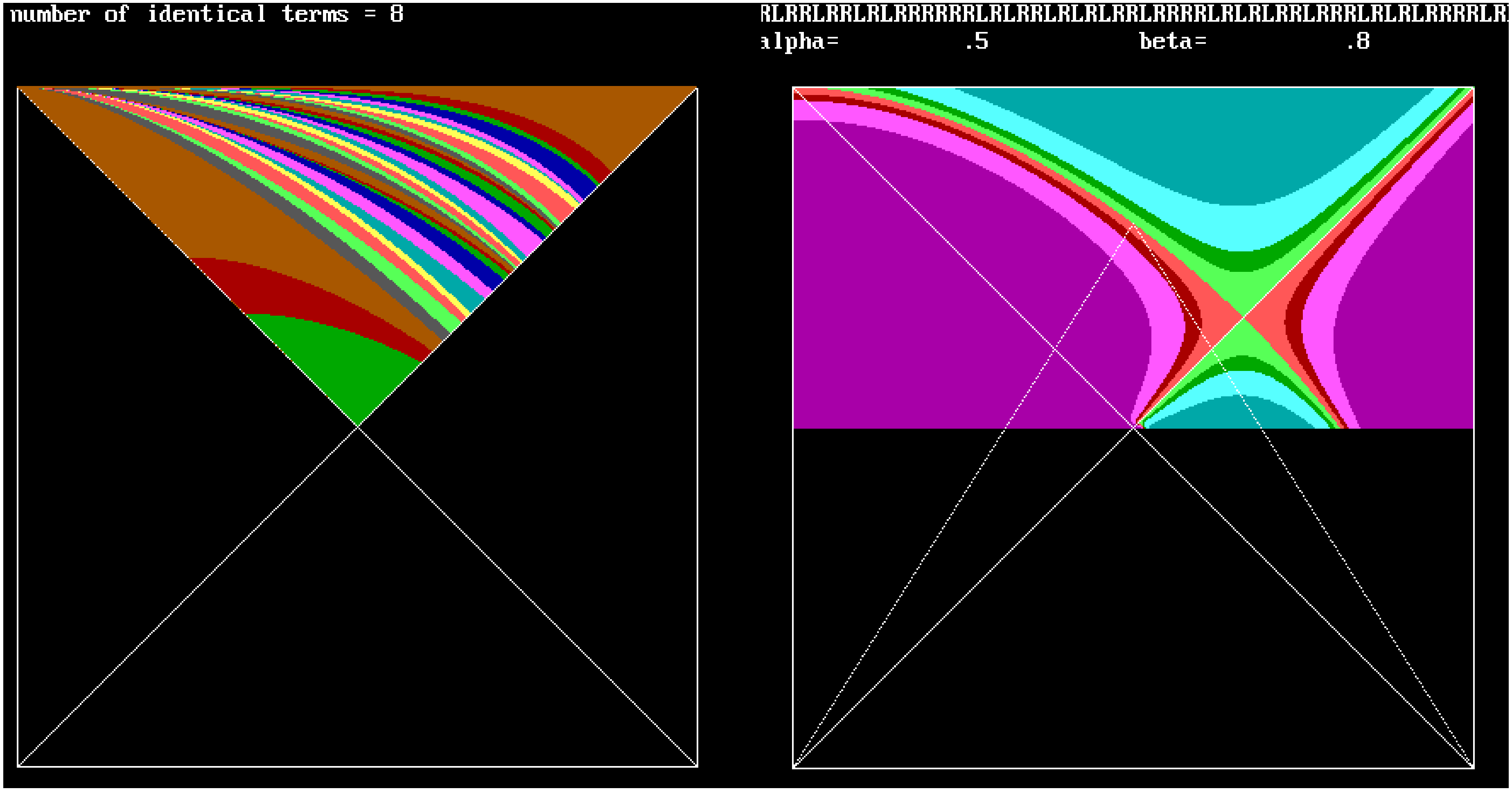

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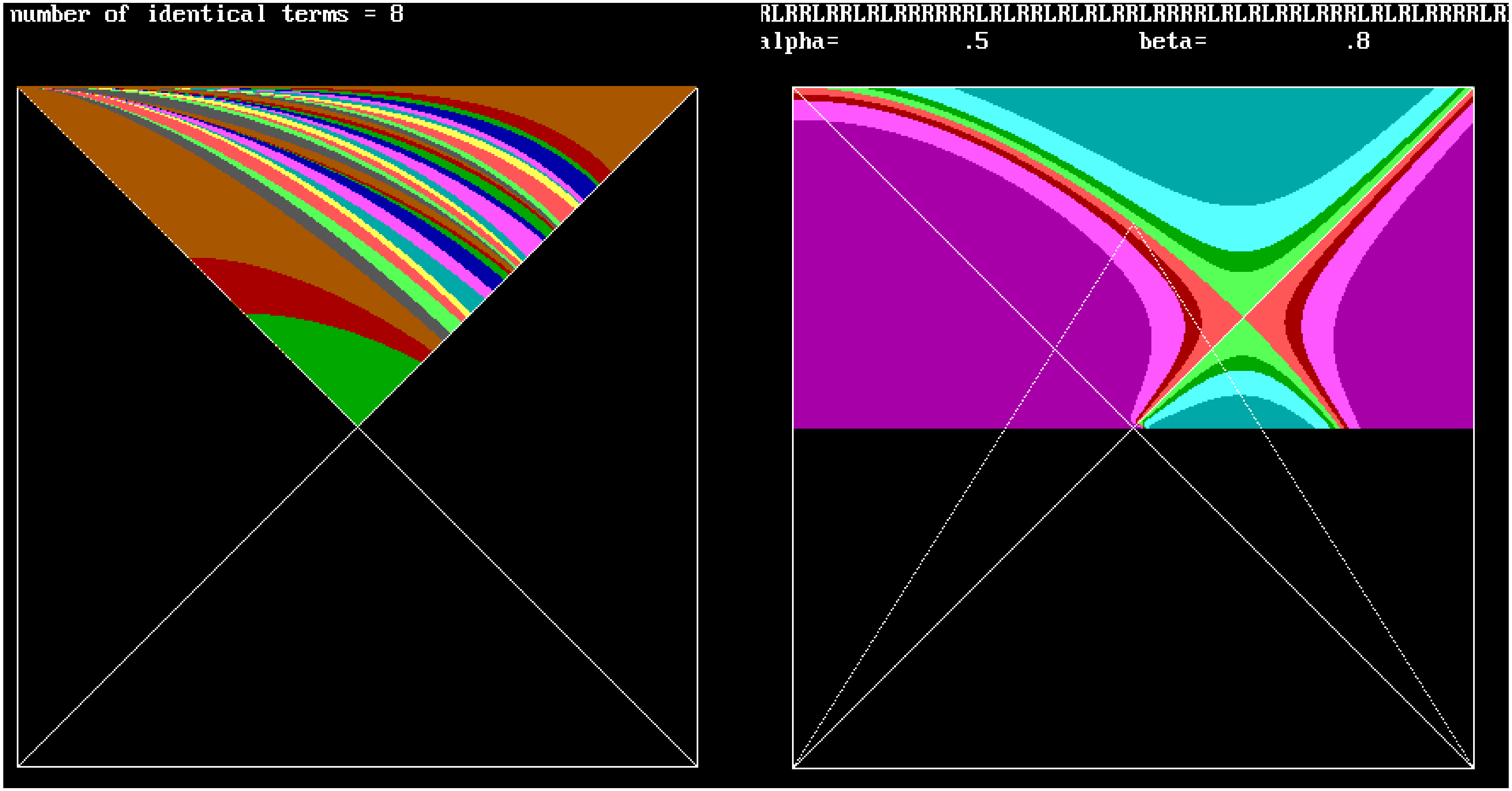

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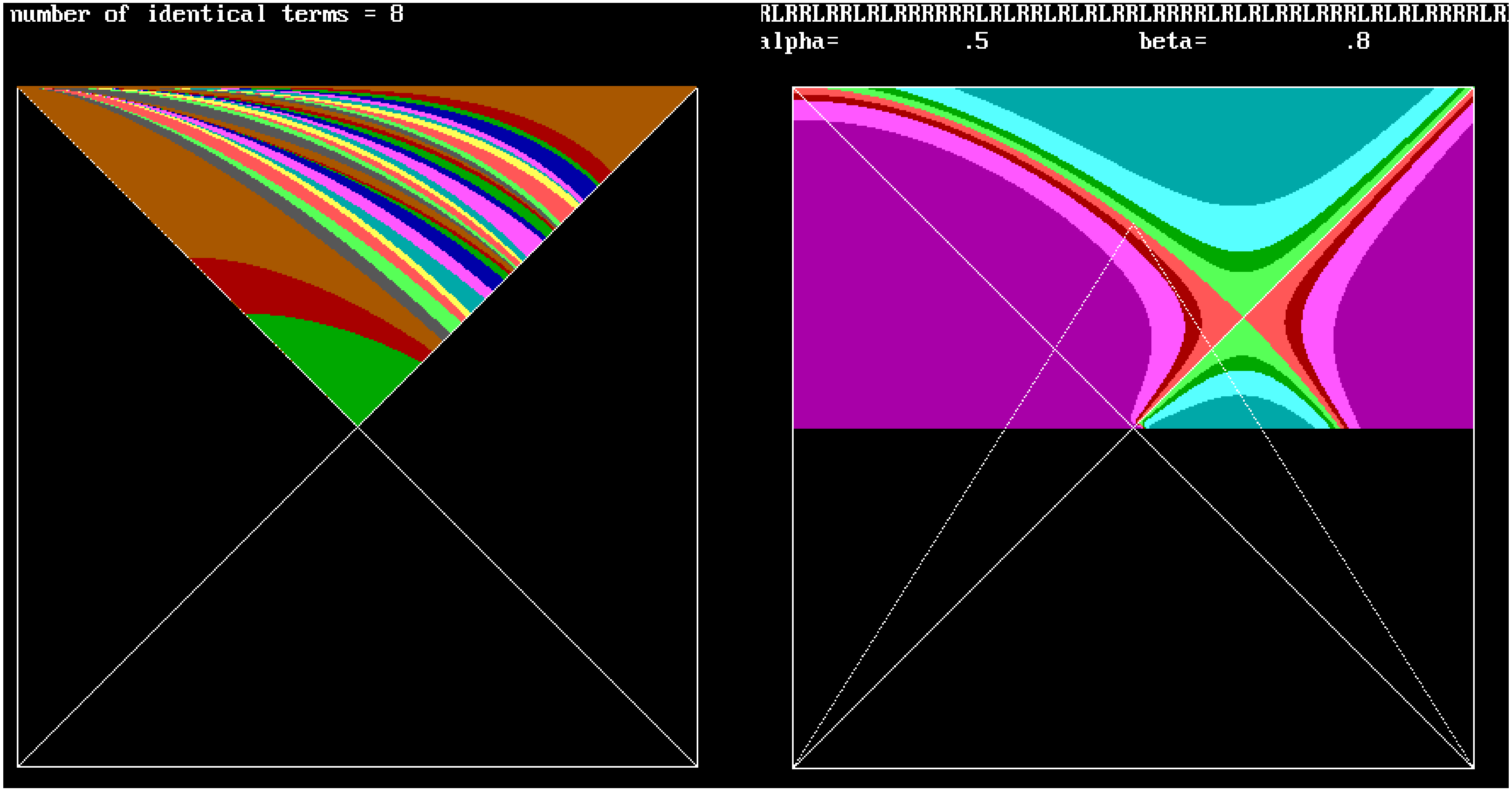
Then $\Theta^{\beta_0}(\beta, \beta) = 0$ *for all* $\beta \in (\frac{1}{2}, 1)$ $\partial_\alpha \Theta^{\beta_0}(\beta_0, \beta_0) = 0, \; \partial_\beta \Theta^{\beta_0}(\beta_0, \beta_0) = 0.$



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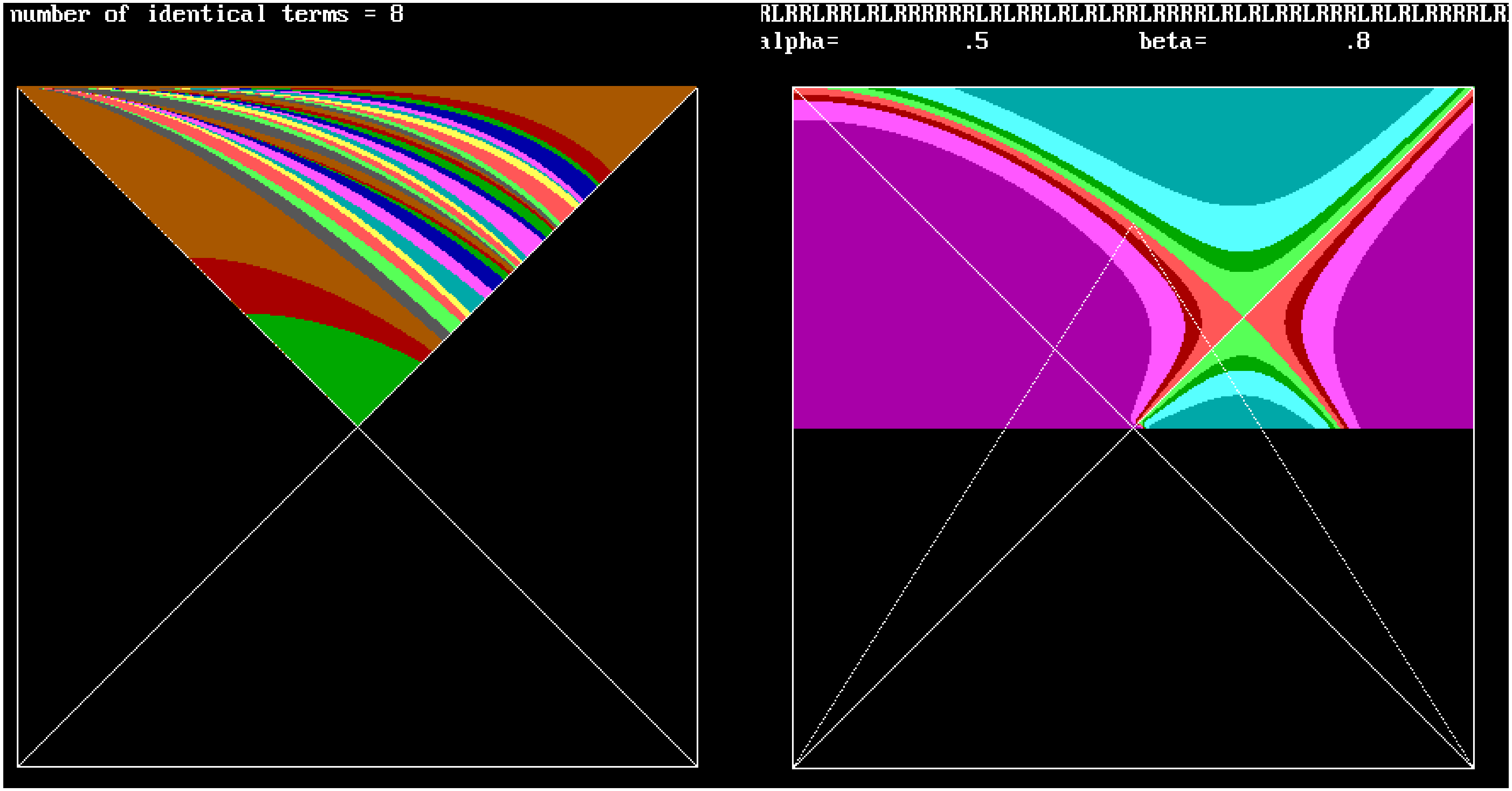

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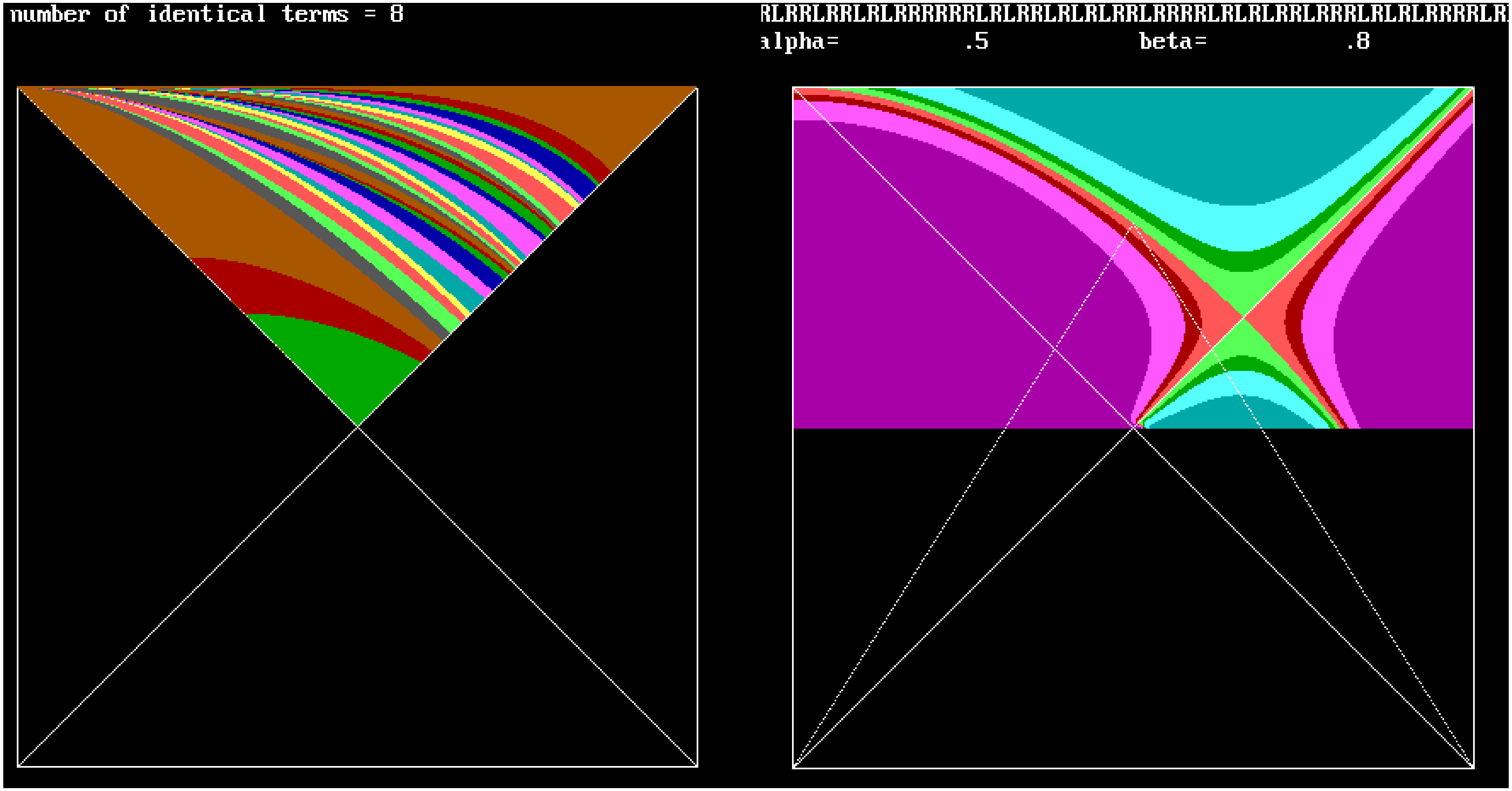

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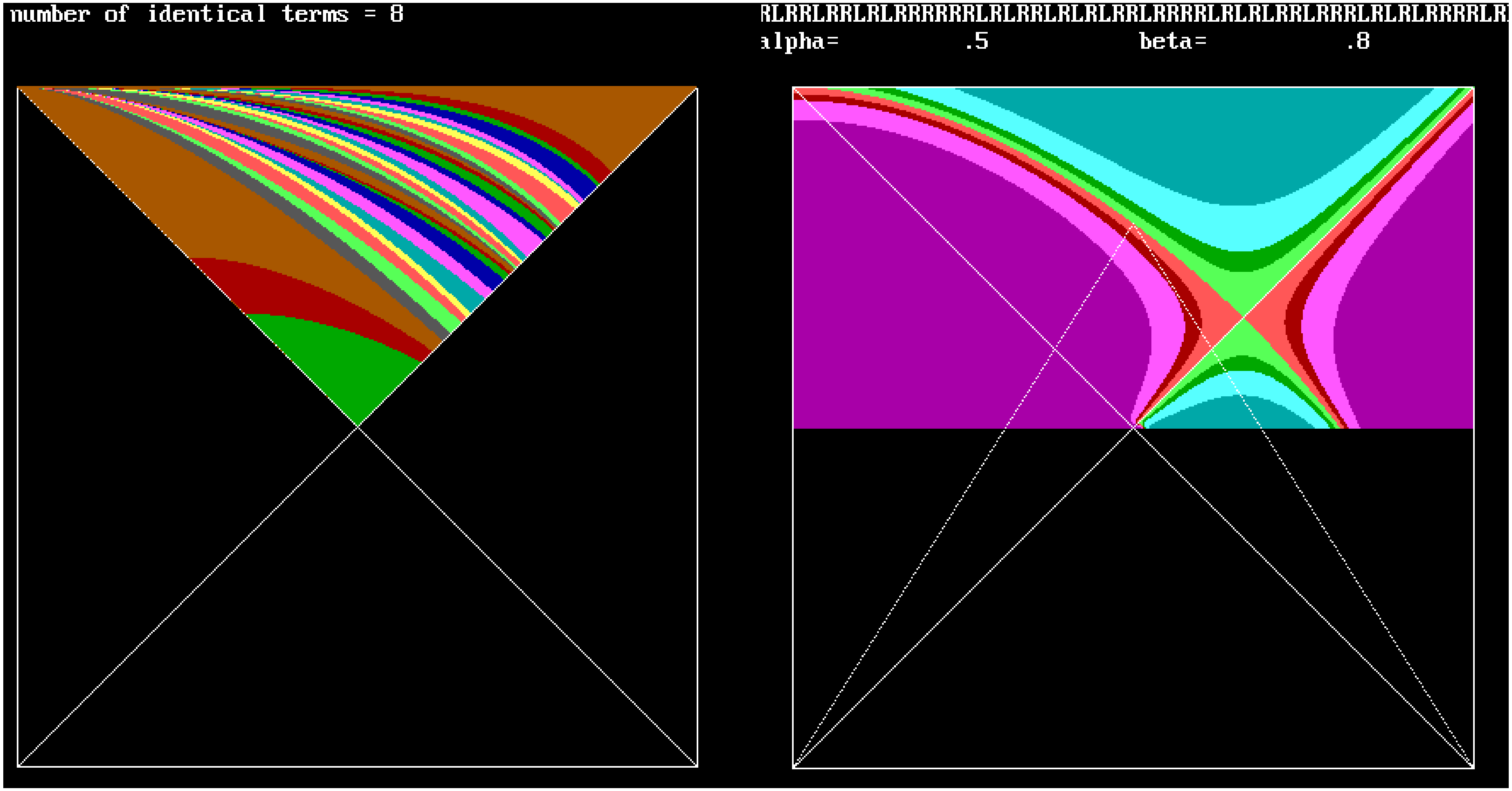

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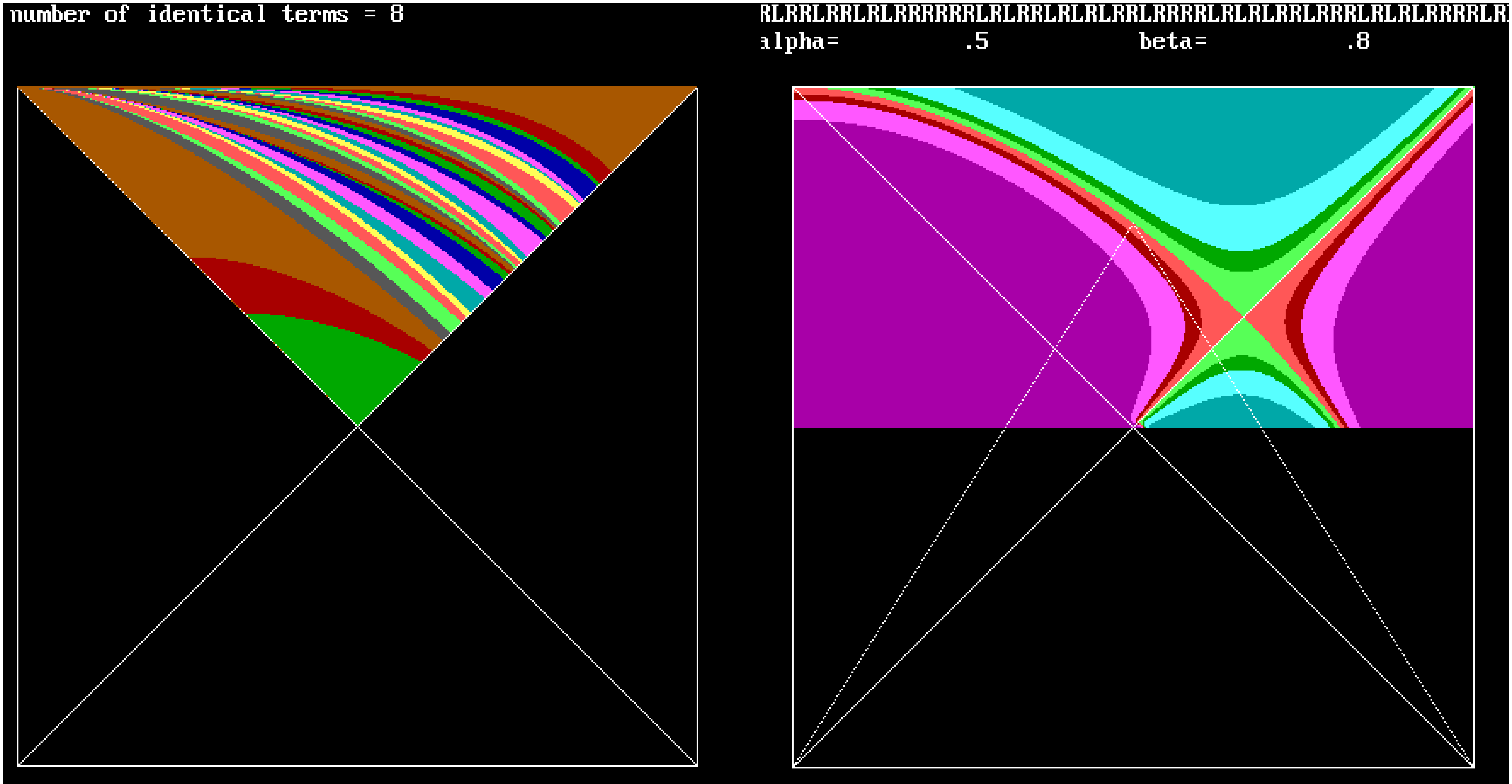

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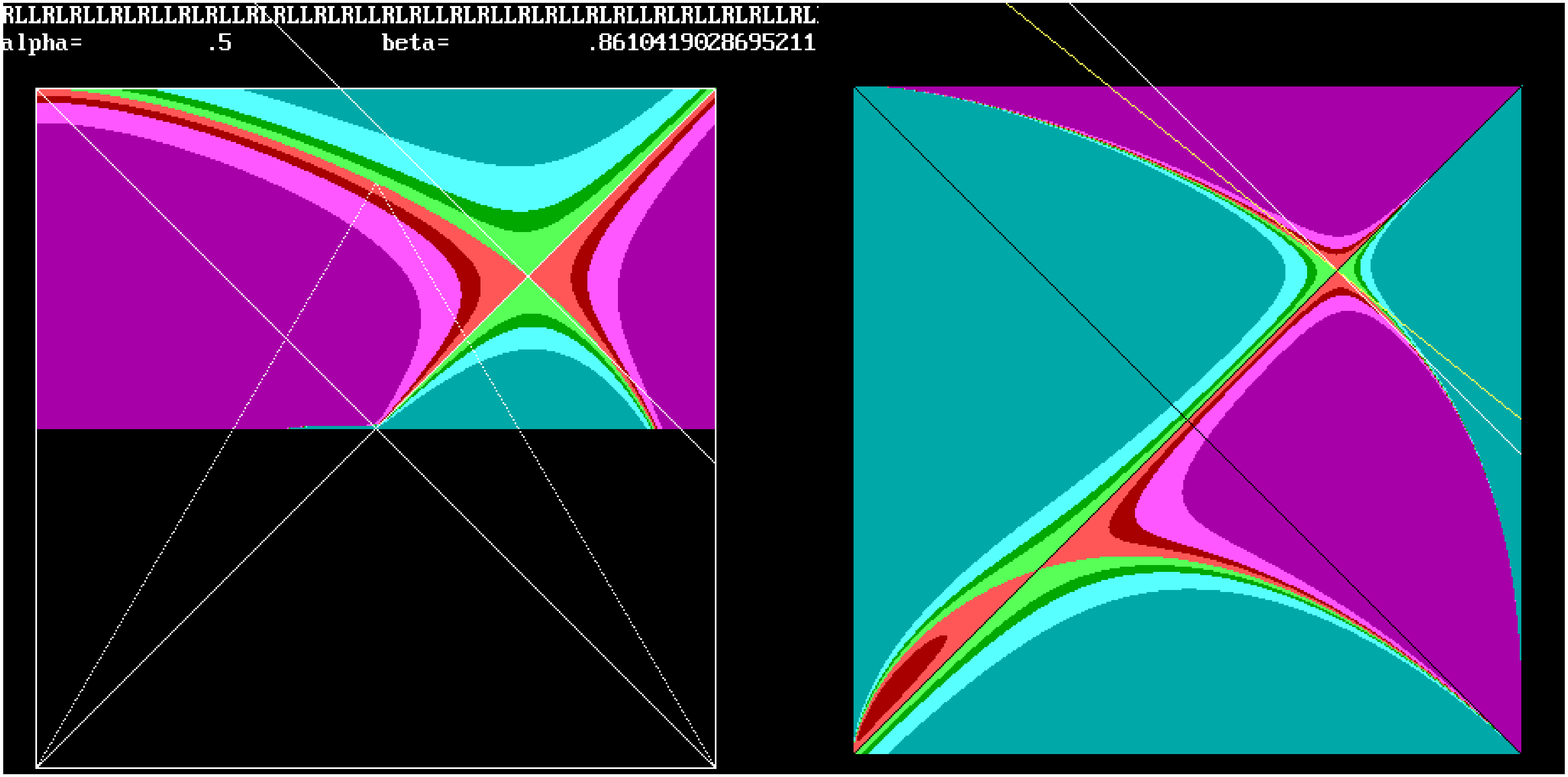

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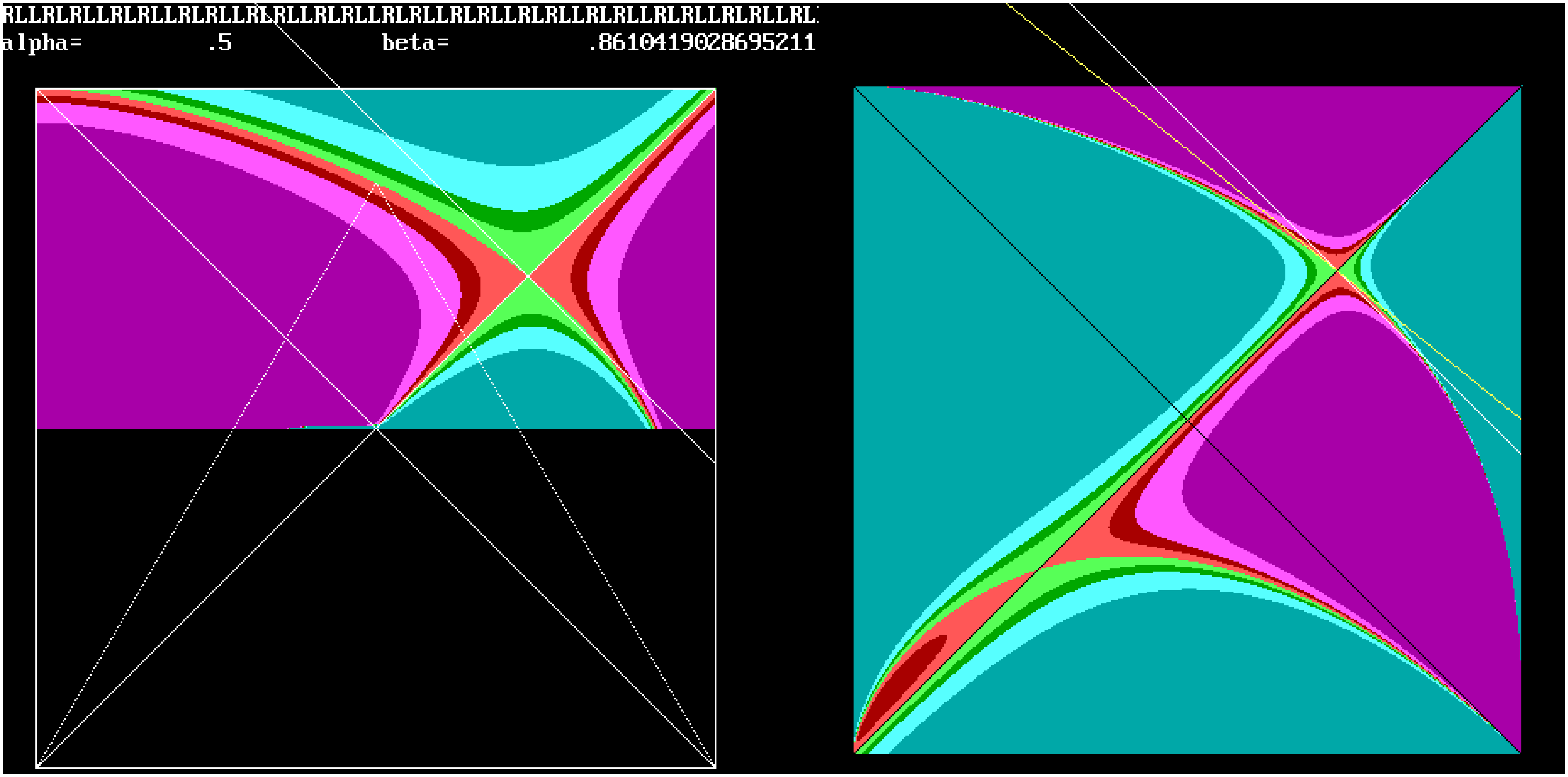

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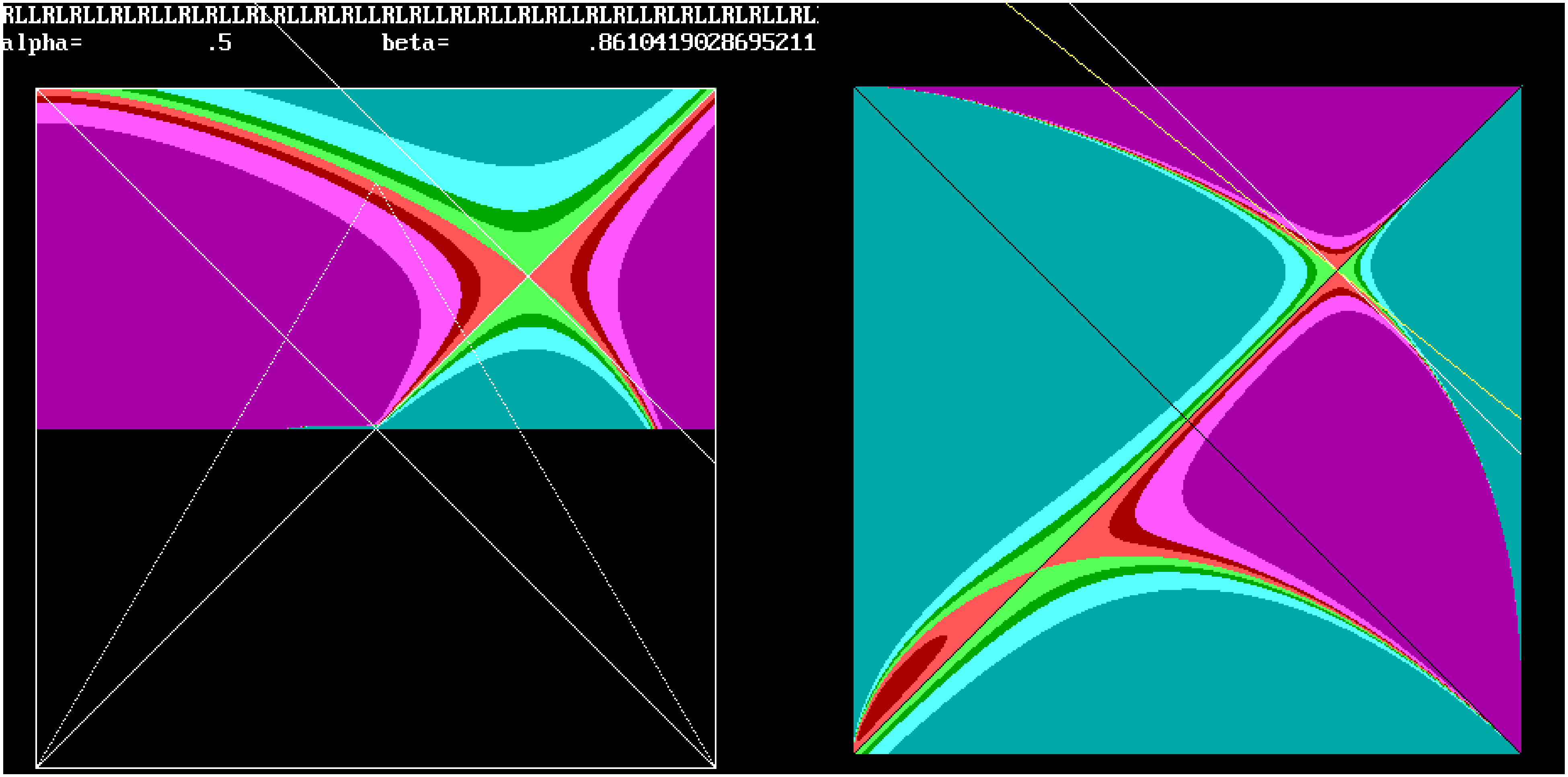

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