Group Colorings and Bernoulli Subflows

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Bernoulli Flows

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$$k^G := \text{ set of all functions } G \to \{0, \ldots, k-1\} = \prod_{g \in G} \{0, 1, \ldots, k-1\}.$$

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A point $x \in k^G$ is *aperiodic* (or *free*) if its stabilizer is trivial. We let $F(k^G)$ denote the *free part* of the action, i.e. the set of all aperiodic (free) points of k^G.

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Lemma (Slaman–Steel, 1988)

There is a decreasing sequence (S_n) of Borel complete sections of $F(k^G)$ such that $\bigcap_n S_n = \varnothing$.

Motivating Question

A strengthening of the Slaman–Steel lemma could lead to improved understanding of hyperfinite equivalence relations.

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If $F(k^G)$ where to contain a compact G-invariant subset, then by compactness the answer to the above question must be 'No.'

Free Subflows

A *subflow* of a Bernoulli flow k^G is a closed (hence compact) subset of k^G which is invariant under the action of G. A subflow $X \subseteq k^G$ is *free* if $X \subseteq F(k^G)$.

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When free subflows are present, the Slaman–Steel marker lemma cannot be strengthened in the manner discussed.

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If the answer is to be 'yes,' a constructive proof is needed.

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This is (essentially) the Morse–Thue sequence.

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Theorem (Glasner–Uspenskij, 2007)

If G is abelian or residually finite and $k \geq 2$, then k^G contains a free subflow.

In their proof they constructed a free continuous action of G on some compact space X and argued that there must be a subflow of k^G factoring onto X. Such a subflow must be free.

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Our construction of k-colorings relies on something we call The Fundamental Method.

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Using the undefined points of c, one can encode data and extend c to $c' \in k^G$. The properties above imply that this data is uniquely readable from c'.

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7. We are done if $gs_n u \notin \Delta_n$.
8. If $gs_n u \in \Delta_n$, then gu and $gs_n u$ are close and hence have distinct L_n values.
9. Since the L_n's can be "decoded" from x, there is a small $v \in G$ with $x(gs_n uv) \neq x(guv)$.

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For $A \subseteq G$ the following are equivalent:

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The above theorem says in particular that if $A \subseteq G$ is finite and $y : A \to \{0, \dots, k-1\}$ is any function, then there are continuum-many k-colorings which extend y and have pairwise disjoint orbit closures.

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Descriptive Complexity

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We call G *flecc* if there is a finite $A \subseteq G \setminus \{1_G\}$ such that every $g \in G$ has a power which is conjugate to an element of A, i.e.

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If G is flecc then the set of k-colorings is Σ_2^0-complete. If G is not flecc then the set of k-colorings is Π_3^0-complete.

The Topological Conjugacy Relation

Two subflows $X, Y \subseteq k^G$ are *topologically conjugate* if there is a homeomorphism $\phi : X \to Y$ such that $\phi(g \cdot x) = g \cdot \phi(x)$ for every $g \in G$ and every $x \in X$. Let $TC(k^G)$ denote the topological conjugacy relation on subflows of k^G.

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Recall that an equivalence relation is *countable* if every equivalence class is countable.

Lemma (Clemens, 2009)

TC and its restrictions to $S_F(k^G)$ and $S_{MF}(k^G)$ are countable Borel equivalence relations.

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For two Borel equivalence relations E and F on X and Y, we say E is *reducible* to F if there is a Borel function $f : X \to Y$ such that $x_1 \ E \ x_2 \iff f(x_1) \ F \ f(x_2)$. Intuitively, F is more complicated than E if E is reducible to F.

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Theorem (Gao–Jackson–S; Clemens)

If G is infinite and locally finite, then $TC(k^G)$ and its restriction to $S_F(k^G)$ are non-smooth and hyperfinite. If G is not locally finite, then $TC(k^G)$ and its restriction to $S_F(k^G)$ are universal countable Borel equivalence relations.

The Topological Conjugacy Relation

Theorem (Gao–Jackson–S)

For any countably infinite group G, the restriction of $TC(k^G)$ to $S_{MF}(k^G)$ is non-smooth.

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Open Question

What is the complexity of the restriction of $TC(k^{\mathbb{Z}})$ to $S_{MF}(k^{\mathbb{Z}})$?