Point realizations of Boolean actions

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Outline of Topics

1. Main problem
2. Some answers
3. Groups of isometries and unifying results
4. The borderline case: $C(M, \mathbb{T})$

All metric and topological spaces are assumed to be second countable.

Main problem

X a standard Borel space, for example, $X = [0, 1]$

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f has a **point realization** F.

Topology on $\mathrm{Aut}(\mu)$ = the weakest topology making all the functions

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continuous, where $a, b \in \mathrm{Borel}/\mu$.

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continuous, where $a, b \in \mathrm{Borel}/\mu$.

This is a **Polish group** (separable, completely metrizable) topology on $\mathrm{Aut}(\mu)$.

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We will write $g(a)$ for $\phi(g)(a)$.

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With notation $\phi\colon G \to \mathrm{Aut}(\mu)$ and $\alpha\colon G \times X \to X$, the above equality says

$$\phi(g)(A/\mu) = \{\alpha(g, x)\colon x \in A\}/\mu.$$

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each continuous homomorphism $G \to \mathrm{Aut}(\mu)$ (Boolean action) has a point realization?

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Some answers

The bad side

Some Polish groups have Boolean actions without point realizations.

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Glasner–Tsirelson–Weiss '05:
$G =$ measure classes of measurable functions $[0, 1] \to \mathbb{T}$ with pointwise addition as group operation and with convergence in measure

Becker's example:

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$$a.(d_1, d_2) =$$

where $a \in G$ and $d_i \in \mathrm{Borel}([0,1] \times \{i\})/\mu$, $i = 1, 2$.

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$$a.(d_1, d_2) = ((d_1 \setminus a) \cup (d_2 \cap a), (d_2 \setminus a) \cup (d_1 \cap a)),$$

where $a \in G$ and $d_i \in \mathrm{Borel}([0,1] \times \{i\})/\mu$, $i = 1, 2$.

The good side

Recall that S_∞ is the group of all permutations of $\mathbb{N}$ with composition and the topology of pointwise convergence.

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The proofs of these two results were very different.

Groups of isometries and unifying results

Groups of isometries

X a metric space

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G is a **Polish group of isometries of** X if G is a subgroup of $\mathrm{Iso}(X)$ as a topological group.

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Examples: locally compact groups, closed subgroups of S_∞, **closed subgroups of countable products of locally compact groups**

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The unifying result

Theorem (Kwiatkowska–S.)

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The result unifies the theorems of Mackey and Glasner–Weiss.

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New cases: closed subgroups of countable products of locally compact groups.

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Theorem (Kwiatkowska–S.)

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Proofs

An outline of the proof of the second theorem

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G is an isometry group of a locally compact metric space if and only if G is a **closed subgroup** of a **countable product** of groups of the form

$$S_\infty \ltimes H^{\mathbb{N}},$$

where H is locally compact and S_∞ acts by homomorphisms on $H^{\mathbb{N}}$ by permuting coordinates.

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Lemma

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Lemma

(i) *$S_\infty \ltimes H^{\mathbb{N}}$, where H is locally compact, has $(*)$.*

(ii) *Condition $(*)$ is preserved under taking countable products.*

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Proof uses Yamabe's theorem connecting locally compact groups with Lie groups (Hilbert's 5-th problem)

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Proof uses Yamabe's theorem connecting locally compact groups with Lie groups (Hilbert's 5-th problem) and well behaved dimension on Lie groups.

The borderline case: $C(M, \mathbb{T})$

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$C([0,1], \mathbb{T})$ lies exactly between
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groups with property $(*)$ (which have point realizations).

It does not have "concentration of measure."

Theorem (Moore–S.)

Let M be a compact uncountable metric space. The group $C(M, \mathbb{T})$ has a whirly Boolean action

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Let M be a compact uncountable metric space. The group $C(M, \mathbb{T})$ has a whirly Boolean action (so a Boolean action without a point realization).

An outline of proof in the case $M = 2^{\mathbb{N}}$

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Identify $\mathbb{C}$ with $\mathbb{R}^2$.

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Let γ be the standard Gaussian measure on $\mathbb{C}$ with density

$$\frac{1}{2\pi} e^{-\frac{1}{2}(x_0^2 + x_1^2)}$$

Note

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$$\pi\colon \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \to \frac{z_1 + z_2}{\sqrt{2}} \in \mathbb{C}$$

is **measure preserving** if $\mathbb{C} \times \mathbb{C}$ is taken with $\gamma \times \gamma$ and $\mathbb{C}$ with γ, Note γ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$,

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is **measure preserving** if $\mathbb{C} \times \mathbb{C}$ is taken with $\gamma \times \gamma$ and $\mathbb{C}$ with γ, and

$$\iota\colon \mathbb{T} \ni z \to (z, z) \in \mathbb{T} \times \mathbb{T}$$

is a **continuous embedding**.

$$\begin{array}{c}(\mathbb{C}, \gamma)\\ \Big\uparrow \\ \mathbb{T}\end{array}$$

$$\begin{array}{ccc}
(\mathbb{C}, \gamma) & \xleftarrow{\ \pi\ } & (\mathbb{C}^2, \gamma^2) \\
\uparrow & & \uparrow \\
\mathbb{T} & \xrightarrow{\ \iota\ } & \mathbb{T}^2
\end{array}$$

$$\begin{array}{ccccc} (\mathbb{C},\gamma) & \xleftarrow{\pi} & (\mathbb{C}^2,\gamma^2) & \xleftarrow{\pi^2} & (\mathbb{C}^4,\gamma^4) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{T} & \xrightarrow{\iota} & \mathbb{T}^2 & \xrightarrow{\iota^2} & \mathbb{T}^4 \end{array}$$

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\end{array}$$

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\begin{array}{ccccc}
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We get a Boolean action of $C(2^{\mathbb{N}}, \mathbb{T})$ on the probability measure space $(\mathbb{C}^\infty, \gamma^\infty)$.

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If $a \in \mathbb{R}$ and $B \subseteq \mathbb{R}^{\mathbb{N}}$ is a Borel set of positive $\gamma^{\mathbb{N}}$-measure, then

$$\gamma^{\mathbb{N}}(\sqrt{1+a^2}B + ay) > 0, \quad \text{for } \gamma^{\mathbb{N}}\text{-a.e. } y \in \mathbb{R}^{\mathbb{N}}.$$

Question.

Question. Characterize Polish groups with the point realization property?