The hereditarily ordinal definable sets in models of determinacy

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Plan:

I. Set theory as a foundation for mathematics.
II. Models of AD, and their HOD's.
III. Pure extender models.
IV. HOD^M as a mouse.

Set theory as a foundation

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(1) ZFC. (Cantor, Zermelo,...1870–1930.)
(2) Applications to the study of the reals. *Descriptive set theory*.(Borel, Baire, Lebesgue, Lusin,...1900–1930.)
(3) ZFC is incomplete. (Gödel 1937, Cohen 1963, ...) Even in the relatively concrete domain of descriptive set theory.

Expanded answer: ZFC plus large cardinal hypotheses.

Theorem (Solovay 1966, Martin 1968)

Assume there is a measurable cardinal; then

(1) *All $\utilde{\Sigma}^1_2$ sets of reals are Lebesgue measurable.*

(2) *All $\utilde{\Pi}^1_1$ sets of irrationals are determined.*

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Remarks.

(a) The measurable cardinal is needed here.

(b) (2) implies (1).

Determinacy

Let $A \subseteq \omega^\omega$. ($\omega^\omega = \mathbb{R} =$ "the reals".) G_A is the infinite game of perfect information: players I and II play $n_0, n_1, n_2, \ldots$, alternating moves. I wins this run iff $\langle n_i \mid i < \omega \rangle \in A$.

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Definition

(1) A set $A \subseteq \omega^\omega$ is *determined* iff one of the players in G_A has a winning strategy.

(2) Γ *determinacy* is the assertion that all $A \in \Gamma$ are determined. AD is the assertion that all $A \subseteq \omega^\omega$ are determined.

ZFC proves there are non-determined A. The proof uses the axiom of choice.

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If there are arbitrarily large Woodin cardinals, then $L(\mathbb{R}) \models \mathrm{AD}$.

The fact that $L(\mathbb{R}) \models \mathrm{AD}$ is the basis of a detailed structure theory for $L(\mathbb{R})$. (Due to many people, 1960s onward.)

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    (a) natural numbers,
    (b) reals,
    (c) *nice* sets of reals.

(2) Nothing like our current large cardinal hypotheses decides CH, or various other natural questions about *arbitrary* sets of reals.

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   (a) the canonical inner models (like Gödel's universe L of constructible sets), and
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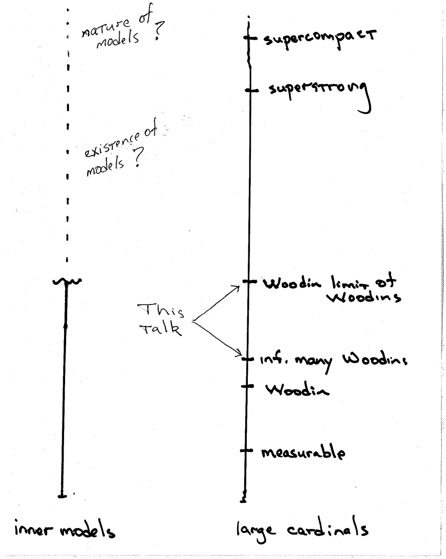
(4) *Inner model program*: associate to each large cardinal hypothesis a canonical, minimal universe in which the hypothesis holds true, and analyze that universe in detail.

(5) In the region we understand, there are three intertwined types of model "at the center":

  (a) canonical models of AD,
  (b) their HOD's,
  (c) pure extender models.

(The *triple helix*.)



nature of models ?
existence of models ?
This talk
supercompact
superstrong
Woodin limit of Woodins
inf. many Woodins
Woodin
measurable
inner models
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Homogeneously Suslin sets

Definition

A set $A \subseteq \omega^\omega$ is Hom_∞ iff for any κ, there is a continuous function $x \mapsto \langle (M_n^x, i_{n,m}^x) \mid n, m < \omega \rangle$ on ω^ω such that for all x, $M_0^x = V$, each M_n^x is closed under κ-sequences, and

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Theorem (Martin, S., Woodin 1985)

If there are arbitrarily large Woodin cardinals, then for any pointclass Γ properly contained in Hom_∞, every set of reals in $L(\Gamma, \mathbb{R})$ is in Hom_∞, and thus $L(\Gamma, \mathbb{R}) \models AD^+$.

Generic absoluteness

A $(\Sigma^2_1)^{Hom_\infty}$ *statement* is one of the form:
$\exists A \in Hom_\infty(V_{\omega+1}, \in, A) \models \varphi$.

Theorem (Woodin)

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Ordinal definability

A set is OD just in case it is definable over the universe of sets from ordinal parameters.

A set is in HOD just in case it, all its members, all members of members, etc., are OD.

Theorem (Gödel, late 30s?)

Assume ZF*; then HOD* $\models$ ZFC*.*

Conjecture. Assume there are arbitrarily large Woodin cardinals, and let $\Gamma \subsetneq Hom_\infty$ be a boldface pointclass; then $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models \mathrm{GCH}$.

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The true goal is to develop a fine structure theory for HOD^M, where $M \models \mathrm{AD}^+$. It is unlikely that one could prove the conjectures without doing that.

Such a theory has been developed for M below the minimal model of $\mathrm{AD}_\mathbb{R}$ + "Θ is regular."

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Definition (Suslin representations)

Let $A \subseteq \mathbb{R}$ and $\kappa \in \mathrm{OR}$; then A is κ-*Suslin* iff there is a tree T on $\omega \times \kappa$ such that $A = p[T] = \{x \mid \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T\}$.

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Theorem (Woodin, late 80's)

Assume AD^+*; then*

(a) *Every* Σ^2_1 *set is Suslin via an ordinal definable tree.*

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(b) *Suppose* $\exists A \subseteq \mathbb{R}(V_{\omega+1}, \in, A) \models \varphi$*; then there is a* Δ^2_1 *set* A *such that* $(V_{\omega+1}, \in, A) \models \varphi$*.*

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(c) *Suppose* $\exists A \subseteq \mathbb{R}(V_{\omega+1}, \in, A) \models \varphi$*; then* $HOD \models (\exists A \subseteq \mathbb{R}(V_{\omega+1}, \in, A) \models \varphi)$.

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Thus Σ^2_1 truths about the AD^+ world go down to its HOD. Since HOD $\models$ "there is a wellorder of the reals", they don't go up. But HOD^M can see M, as an inner model of a generic extension of itself.

The Solovay sequence

Definition

(AD^+.) For $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal α such that there is no surjection of $\mathbb{R}$ onto α which is ordinal definable from A and a real. We set

$$\begin{aligned}
\theta_0 &= \theta(\emptyset), \\
\theta_{\alpha+1} &= \theta(A), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha, \\
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$\theta_{\alpha+1}$ is defined iff $\theta_\alpha < \Theta$. Note $\theta(A) < \Theta$ iff there is some $B \subseteq \mathbb{R}$ such that $B \notin \mathrm{OD}(\mathbb{R} \cup \{A\})$. In this case, $\theta(A)$ is the least Wadge rank of such a B.

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$L(\mathbb{R}) \models \theta_0 = \Theta$.

Theorem (Woodin, mid 80's)

Assume AD^+*, and suppose* A *and* $\mathbb{R} \setminus A$ *are Suslin; then*

(a) *All* $\Sigma^2_1(A)$ *sets of reals are Suslin, and*

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Theorem (Martin, Woodin, mid 80's)

Assume AD^+*; then the following are equivalent:*

(1) $\mathsf{AD}_{\mathbb{R}}$,

(2) *Every set of reals is Suslin,*

(3) $\Theta = \theta_\lambda$*, for some limit* λ.

Theorem (Woodin late 90s)

If there are arbitrarily large Woodin limits of Woodin cardinals, then for some $\Gamma \subsetneq \mathit{Hom}_\infty$, $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R}$.

In fact $\mathsf{AD}_\mathbb{R}$ is weaker than a Woodin limit of Woodins. Its exact consistency strength is known.
The computation uses the theory of HOD^M, for $M \models \mathsf{AD}^+$.

Large cardinals in HOD

Theorem

Assume AD; *then*

(a) *Θ is a limit of measurable cardinals (Solovay, Moschovakis, late 60's).*

(b) *Every measure on a cardinal $< \Theta$ is ordinal definable (Kunen, early 70's).*

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whenever $\beta = 0$ or β is a successor ordinal.

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Corollary (Woodin, late 80's)

Assume $\mathrm{AD}_\mathbb{R}$*; then θ is a limit of cardinals that are Woodin in HOD.*

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Definition

A *premouse* is a structure of the form $\mathcal{M} = (J_\gamma^{\vec{E}}, \in, \vec{E})$, where $\vec{E}$ is a *coherent sequence of extenders*.

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Remark. The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

Proper class premice are sometimes called extender models.

Coherence: for all $\alpha \leq \gamma$, $E_\alpha = \emptyset$, or E_α is an extender (system of ultrafilters) with support α over $\mathcal{M}|\alpha = (J_\alpha^{\vec{E}\restriction\alpha}, \in, \vec{E} \restriction \alpha)$ coding

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such that

$$i(\vec{E} \restriction \alpha) \restriction \alpha = \vec{E} \restriction \alpha \text{ and } i(\vec{E} \restriction \alpha)_\alpha = \emptyset.$$

The iteration game

A *mouse* is an iterable premouse.

Let $\mathcal{M}$ be a premouse. In $\mathcal{G}(\mathcal{M}, \theta)$, players I and II play for θ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$ at the base.

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Round $\beta + 1$*:* I picks an extender E_β from the sequence of $\mathcal{M}_\beta$, and $\xi \leq \beta$. We set

$$\mathcal{M}_{\beta+1} = \mathrm{Ult}(\mathcal{M}_\xi, E_\beta),$$

I must choose ξ so that this ultrapower makes sense.

Round λ*, for* λ *limit:* II picks a branch b of $\mathcal{T}$ which is cofinal in λ, and we set

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As soon as an illfounded model $\mathcal{M}_\alpha$ arises, player I wins. If this has not happened after θ rounds, then II wins.

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Theorem (Comparison Lemma, Kunen 1970, Mitchell-S. 1989)

Let Σ and Γ be $\omega_1 + 1$ iteration strategies for countable premice $\mathcal{M}$ and $\mathcal{N}$ respectively. Then either

(a) *there is a Γ-iterate $\mathcal{P}$ of $\mathcal{N}$, and a map $j\colon \mathcal{M} \to \mathcal{P}|\eta$ produced by Σ-iteration,or*

(b) *there is a Σ-iterate $\mathcal{P}$ of $\mathcal{M}$, and a map $j\colon \mathcal{N} \to \mathcal{P}|\eta$ produced by Γ-iteration.*

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Corollary

If $\mathcal{M}$ is an ω_1+1-iterable premouse, and $x\in\mathbb{R}\cap\mathcal{M}$, then x is ordinal definable.

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(AD^+) *Mouse Capturing* (MC) is the statement: for any reals x, y, the following are equivalent:

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Mouse Set Conjecture: Assume AD^+, and that there is no ω_1-iteration strategy for a mouse with a superstrong cardinal; then Mouse Capturing holds.

Remark. Assume AD^+. Mouse capturing is then equivalent to: whenever x is a real, and

$$\exists A (V_{\omega+1}, \in, A) \models \varphi[x]$$

is a true Σ^2_1 statement about x, then there is an ω_1-iterable mouse M over x such that

$$M \models \text{ ZC + ``there are arbitrarily large Woodin cardinals''},$$

and

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Remark. Assume AD^+. Mouse capturing is then equivalent to: whenever x is a real, and

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is a true Σ_1^2 statement about x, then there is an ω_1-iterable mouse M over x such that

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and

$$M \models \exists A \in Hom_\infty(V_{\omega+1}, \in, A) \models \varphi[x].$$

That is, Σ_1^2 truth is captured by mice.

HOD^M as a mouse

Theorem (Woodin, S. early 90s)

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What is the full HOD$^{L(\mathbb{R})}$? A new species of mouse!

Let M_ω be the canonical minimal extender model with ω Woodins, and Σ its *unique* iteration strategy. Then

$$\mathrm{HOD}^{L(\mathbb{R})} = L[N, \Lambda],$$

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(Woodin, 1995.) The iteration strategy Λ is new canonical information. (No iterable extender model with a Woodin knows how to iterate itself for iteration trees based on its bottom Woodin.) Nevertheless, Λ adds no new bounded subsets of Θ beyond those already in N, and it preserves the Woodinness of Θ.

HOD-mice

Work of Woodin (late 90s) and Sargsyan (2008) led to an analysis of HOD^M as a *hod-mouse*, for $M \models \mathrm{AD}^+$ up to the minimal model of $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular. In such M:

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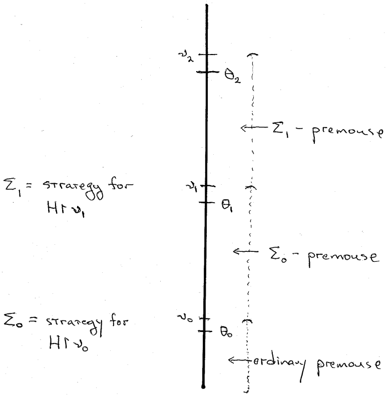

$$H = HOD^M$$

for $M \models AD^+$

Some equiconsistencies

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(1) AD^+ models,

(2) their HOD's,

(3) pure extender models.

The core model induction

One proves consistency strength lower bounds by climbing all three staircases together.

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The following are equiconsistent

(1) ZFC + *"there is an ω_1-dense ideal on ω_1* + CH + $(*)$,

(2) ZF + $\mathsf{AD}_{\mathbb{R}}$ + *"Θ is regular"*.

Theorem (Sargsyan 2011)

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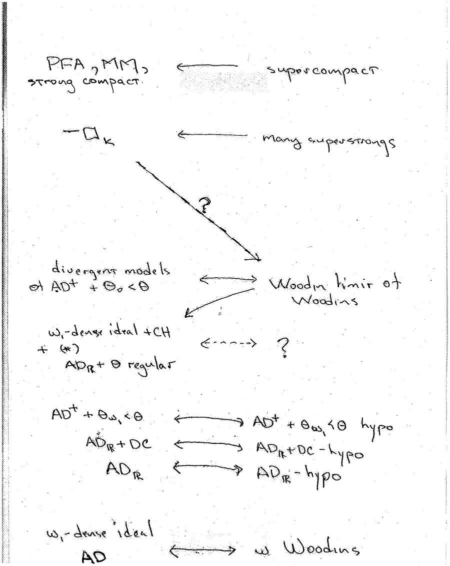
Holy Grail: Con(ZFC + PFA) implies Con(ZFC+ "there is a supercompact cardinal").


PFA, MM,
strong compact
supercompact
¬□κ
many superstrongs
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Beyond $AD_{\mathbb{R}} + \Theta$ regular

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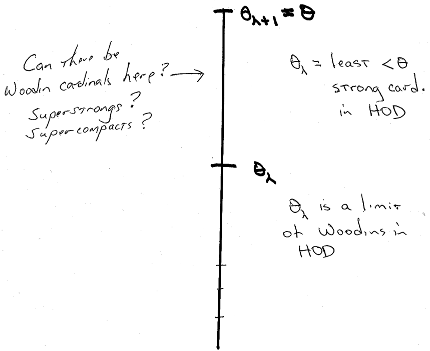
LST implies that for $\Gamma = \{A \mid w(A) < \theta_\lambda\}$, $L(\Gamma, \mathbb{R}) \models \Theta$ is regular.

Current techniques seem likely to lead to: If M is the minimal model of LST, then $\text{HOD}^M \models$ GCH.



HOD^M, for

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Key Question: In the LST situation, can HOD have Woodin cardinals strictly between the largest Suslin cardinal and Θ? Can it have superstrongs, or supercompacts, or... in that interval? If so:

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(1) The comparison problem for hod mice becomes much harder.

(2) A *vision of ultimate L* becomes possible.

Is V a hod mouse?

The following is an axiom recently proposed by Hugh Woodin:

- if
$$\exists \alpha (V_\alpha \models \varphi),$$
then for some $M \models \mathsf{AD}^+$ such that $\mathbb{R} \cup \mathrm{OR} \subseteq M$,
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(c) It may be consistent with all the large cardinal hypotheses.