

The minimal flows of S_∞

Todor Tsankov

Université Paris Diderot – Paris 7

Logic, Dynamics, and Their Interactions, Denton, June 4, 2012

Minimal flows

G — topological group.

A **G -flow**, $G \curvearrowright X$, is a compact Hausdorff space X equipped with a continuous action of G .

Morphisms: if X and Y are G -flows, a **homomorphism** from X to Y is a continuous map $\pi: X \rightarrow Y$ that commutes with the G -actions, i.e.

$$\pi(g \cdot x) = g \cdot \pi(x) \quad \text{for all } x \in X, g \in G.$$

A flow is **minimal** if it has no proper subflows or, equivalently, if every orbit is dense.

Compactness + Zorn's lemma \implies every flow contains a minimal subflow.

Minimal flows are some of the main objects of study in topological dynamics.

The universal minimal flow

For every group G , there exists a **universal minimal G -flow** (a minimal G -flow that maps onto any other minimal G -flow). For example, one can take any minimal subflow of the product

$$\prod \{M : M \text{ is a minimal } G\text{-flow}\}.$$

It is a standard theorem that the universal property characterizes the universal minimal flow up to isomorphism.

Alternatively, if G is discrete, the pointed flow $(\beta G, 1_G)$ is a universal pointed flow for G , i.e. for every flow $G \curvearrowright X$ and every $x_0 \in X$, there exists a homomorphism

$$\pi: \beta G \rightarrow X \quad \text{such that} \quad \pi(1_G) = x_0.$$

Consequently, any minimal subflow of βG is universal for all the minimal flows.

The universal minimal flow (cont.)

We see that in this case, the universal minimal flow is non-metrizable and not amenable to a concrete description.

This is reflected by the large variety of minimal flows that exist for discrete groups. For example, studying (minimal) subflows of the shift $\mathbf{Z} \curvearrowright 2^{\mathbf{Z}}$ is a subject of its own (symbolic dynamics).

If G is compact, then every minimal G -flow is transitive, that is, of the form $G \curvearrowright G/H$, where H is a closed subgroup of G and the universal minimal flow is the left action of G on itself.

Another case in which the situation trivializes is when the group G is **extremely amenable**, i.e. its universal minimal flow is a singleton. This turns out to be the case for many symmetry groups of continuous objects (the infinite-dimensional unitary group, the group of measure-preserving transformations of the interval, etc.)

The universal minimal flow of S_∞

S_∞ is the group of all permutations of the natural numbers, equipped with the pointwise convergence topology: a basis at the identity is given by the **open subgroups**

$$V_n = \{g \in S_\infty : g \cdot i = i \text{ for all } i = 0, \dots, n-1\}.$$

The set

$$\text{LO} = \{x \in 2^{\mathbf{N} \times \mathbf{N}} : x \text{ is a linear order}\}$$

is a compact subset of $2^{\mathbf{N} \times \mathbf{N}}$ on which S_∞ acts via the **logic action**:

$$a <_{g \cdot x} b \iff g^{-1} \cdot a <_x g^{-1} \cdot b, \quad g \in S_\infty, x \in \text{LO}, a, b \in \mathbf{N}.$$

The action is minimal: if $U \subseteq \text{LO}$ is the open set $0 < 1 < 2 < 3$ and $x \in \text{LO}$ is such that $3 <_x 2 <_x 0 <_x 1$, then there is an obvious permutation g that sends x in U .

The universal minimal flow of S_∞ (cont.)

Theorem (Glasner–Weiss)

The flow $S_\infty \curvearrowright \text{LO}$ is the universal minimal flow of S_∞ .

Let $\eta_0 \in \text{LO}$ be a linear order isomorphic to $(\mathbf{Q}, <)$. For $x \in \text{LO}$,
 $x \in S_\infty \cdot \eta_0$ iff x is isomorphic to η_0 iff

$$\forall a, b \exists c \quad a <_x c <_x b; \quad \text{and}$$

$$\forall a \exists b, c \quad b <_x a <_x c,$$

which shows that the orbit $S_\infty \cdot \eta_0$ is G_δ .

Let H be the stabilizer of η_0 in S_∞ . Then H is isomorphic to $\text{Aut}(\mathbf{Q}, <)$ and

Theorem (Pestov)

The group $\text{Aut}(\mathbf{Q}, <)$ is extremely amenable.

The universal minimal flow of S_∞ (proof)

Say that the homogeneous space G/H is **precompact** if the natural uniformity, whose entourages of the diagonal are

$$\mathcal{U}_V = \{(gH, v gH) : v \in V, g \in G\}, \quad V \text{ is a symmetric nbhd of } 1_G,$$

is precompact. Equivalently, for every V , there exists a finite F such that $VFH = G$.

H — the stabilizer of η_0 in LO . S_∞/H is precompact. It is not difficult to check that

$$(S_\infty \curvearrowright \widehat{S_\infty/H}) \cong (S_\infty \curvearrowright \text{LO}).$$

Let now $S_\infty \curvearrowright X$ be any flow. As H is extremely amenable, there exists $x_0 \in X$ fixed by H . Define a map $\phi: S_\infty/H \rightarrow X$ by $\phi(gH) = g \cdot x_0$. ϕ is uniformly continuous and extends to a map $\hat{\phi}: \text{LO} \rightarrow X$.

Invariant closed equivalence relations

As LO is the universal minimal flow for S_∞ , for any other minimal flow X , there exists a quotient S_∞ -map $\pi: \text{LO} \rightarrow X$. To every such map corresponds an equivalence relation \mathcal{R}_π on LO defined by

$$x \mathcal{R}_\pi y \iff \pi(x) = \pi(y).$$

\mathcal{R}_π is an **invariant, closed equivalence relation, icer** for short. Conversely, any icer gives a quotient of LO.

Classifying the minimal flows of S_∞ therefore boils down to classifying the icers on LO.

It turns out that there are only countably many such icers, each one of them can be generated by a single pair $(x, y) \in \text{LO} \times \text{LO}$, and each quotient can be expressed as the set of models of a universal theory (like the theory of linear orders) and is, in particular, zero-dimensional.

The quotients coming from groups

As LO has a G_δ orbit (that of η_0), it is easy to see that any other minimal S_∞ -flow X must also have a G_δ orbit and that the quotient map $\pi: \text{LO} \rightarrow X$ sends η_0 in that orbit.

Let $x_0 = \pi(\eta_0)$ and let H_{x_0} be the stabilizer of x_0 . Then $H \leq H_{x_0} \leq S_\infty$.

Conversely, any such subgroup H' is co-precompact and there exists a natural quotient map

$$\text{LO} \cong \widehat{S_\infty/H} \rightarrow \widehat{S_\infty/H'}.$$

So for every minimal flow X , we have the factorization

$$\text{LO} \rightarrow \widehat{S_\infty/H_{x_0}} \rightarrow X,$$

where the second map is **almost one-to-one** (one-to-one on a comeager set).

Quotients coming from groups (cont.)

We define certain S_∞ -maps $\pi: \text{LO} \rightarrow 2^{\mathbb{N}^k}$. Then $\pi(\text{LO})$ is a minimal flow of S_∞ .

- ▶ the **betweenness relation (BR)** ($k = 3$)

$$B_x(a, b, c) \iff (a <_x b <_x c) \vee (c <_x b <_x a);$$

- ▶ the **circular order (CO)** ($k = 3$)

$$K_x(a, b, c) \iff (a <_x b <_x c) \vee (b <_x c <_x a) \vee (c <_x a <_x b);$$

- ▶ the **separation relation (SR)** ($k = 3$)

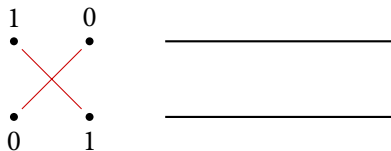
$$S_x(a, b, c, d) \iff (K_x(a, b, c) \wedge K_x(b, c, d) \wedge K_x(c, d, a)) \vee \\ (K_x(d, c, b) \wedge K_x(c, b, a) \wedge K_x(b, a, d))$$

The rest

- ▶ $\text{LO}_{m,n}$ ($k = m + n + 1$)

$$P_{m,n}^x(a_1, \dots, a_m, b, c_1, \dots, c_n) \iff (\bar{a} <_x b <_x \bar{c}) \wedge \left(\bigwedge_{i \neq j} a_i \neq a_j \right) \wedge \left(\bigwedge_{i \neq j} c_i \neq c_j \right).$$

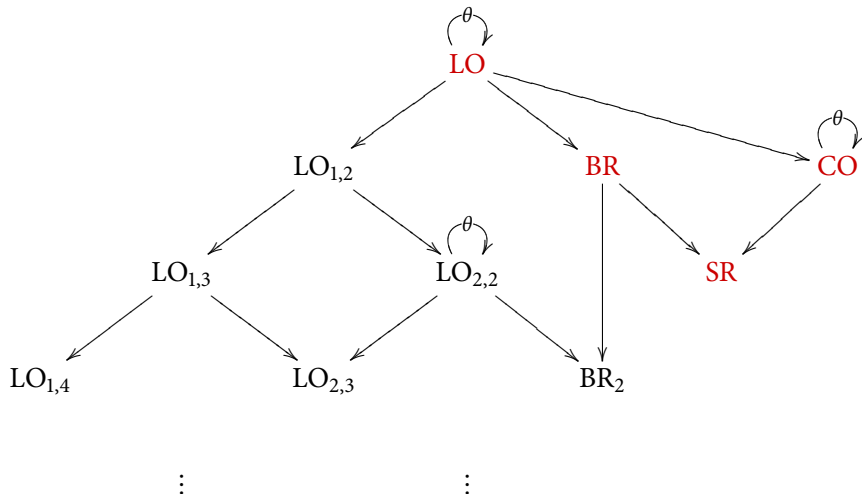
Two linear orders which are identified in $\text{LO}_{2,1}$:



- ▶ $\text{BR}_n = \text{LO}_{n,n}/\text{flip}$ ($k = 2n + 1$)

$$Q_n^x(a_1, \dots, a_n, b, c_1, \dots, c_n) \iff P_{n,n}^x(\bar{a}, b, \bar{c}) \vee P_{n,n}^x(\bar{c}, b, \bar{a}).$$

The complete picture



The arrows represent all possible homomorphisms between the flows.

History

- ▶ Frasnay in 1965 classifies certain sequences of finite groups related to “bi-orders” (sets carrying two linear orders), a classification that basically amounts to the picture above;
- ▶ Cameron in 1976 (unaware of the work of Frasnay) classifies all groups between $\text{Aut}(\mathbf{Q}, <)$ and S_∞ (the red nodes of the diagram);
- ▶ Hodges, Lachlan and Shelah in 1977 independently prove a theorem about indiscernibles that also amounts to a special case of Frasnay’s work.

Some questions

- ▶ What about other automorphism groups?
- ▶ What are some conditions that will ensure that a group of automorphisms has only countably many minimal flows?
- ▶ ...that all minimal flows are zero-dimensional? (An example to be kept in mind is the automorphism group of the random circular order (the G_δ orbit of CO). This group acts minimally on the circle with two orbits.)
- ▶ ...that all minimal flows are realized as the models of theories in a **finite language**?