SOLUTION FOR DECEMBER 2015

Determine if the following series converges:

$$1+\frac{1}{2}\left(\frac{19}{7}\right)+\frac{2!}{3^2}\left(\frac{19}{7}\right)^2+\frac{3!}{4^3}\left(\frac{19}{7}\right)^3+\cdots$$

SOLUTION: The series does indeed converge.

Notice that if we denote $x=\frac{19}{7}$ then we can rewrite the series as:

$$1+\frac{1}{2}x+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\cdots=\sum_{n=0}^{\infty}\frac{n!}{(n+1)^n}x^n.$$

Applying the ratio test we see that:

$$|\frac{a_{n+1}}{a_n}|=\frac{\frac{(n+1)!|x|^{n+1}}{(n+2)^{n+1}}}{\frac{n!|x|^n}{(n+1)^n}}=\frac{(n+1)(n+1)^n|x|}{(n+2)^{n+1}}$$

$$=\frac{(n+1)^{n+1}|x|}{(n+2)^{n+1}}=\left(1-\frac{1}{n+2}\right)^{n+1}|x|=\frac{\left(1-\frac{1}{n+2}\right)^{n+2}}{\left(1-\frac{1}{n+2}\right)}|x|.$$

You might recall from calculus that:

$$\lim_{n\to\infty}(1+\frac{x}{n})^n=e^x$$

and in particular that:

$$\lim_{n\to\infty}(1-\frac{1}{n})^n=e^{-1}.$$

And so it follows that:

$$\lim_{n\to\infty}\frac{\left(1-\frac{1}{n+2}\right)^{n+2}}{\left(1-\frac{1}{n+2}\right)}|x|=e^{-1}|x|.$$

Thus the above series converges when:

$$e^{-1}|x|<1,\ \text{i.e. when } |x|<e$$

and since $x=\frac{19}{7}<e$ then we see that the given series converges by the ratio test. (Note that $\frac{19}{7}=2.714...$ and $e=2.718...$).