Simplify:

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} + \frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} + \frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}}.$$

Solution:
The answer is:

$$\sqrt[3]{4}-1.$$

Here it is helpful to remember the fact that $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$. Letting $x = \sqrt[3]{2}$ and $y = 1$ gives:

$$1 = 2 - 1 = (\sqrt[3]{2})^3 - 1 = (\sqrt[3]{2}-1)(\sqrt[3]{4}+\sqrt[3]{2}+1).$$

Therefore:

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} = \sqrt[3]{2}-1. \tag{1}$$

Next we let $x = \sqrt[3]{3}$ and $y = \sqrt[3]{2}$ and we get

$$1 = 3 - 2 = (\sqrt[3]{3})^3 - (\sqrt[3]{2})^3 = (\sqrt[3]{3}-\sqrt[3]{2})(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}).$$

Therefore:

$$\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} = \sqrt[3]{3}-\sqrt[3]{2}. \tag{2}$$

Finally we let $x = \sqrt[3]{4}$ and $y = \sqrt[3]{3}$ and we get

$$1 = 4 - 3 = (\sqrt[3]{4})^3 - (\sqrt[3]{3})^3 = (\sqrt[3]{4}-\sqrt[3]{3})(\sqrt[3]{16}+\sqrt[3]{12}+\sqrt[3]{9}).$$

Therefore:

$$\frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}} = \sqrt[3]{4}-\sqrt[3]{3}. \tag{3}$$

Finally adding (1)-(3) we see that:

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} + \frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} + \frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}} = \sqrt[3]{4}-1.$$