SOLUTION FOR FEBRUARY 2015

Denote $p = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ and $q = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$. Express:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(i+j)^3}$$

in terms of p and q.

SOLUTION:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(i+j)^3} = p - q$$

We first note that this sum converges absolutely by the integral test and so it follows from a theorem in analysis that we can write out the terms in any order whatsoever and the sum will be the same.

If we write out the sum in an array we see that:

$$\begin{array}{l}
\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \cdots \\[6pt]
\frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \cdots \\[6pt]
\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \cdots \\[6pt]
\frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \frac{1}{10^3} + \cdots \\
. \\
. \\
.
\end{array}$$

Now we add the terms along the diagonals that go from lower left to upper right and we see that we get:

$$\frac{1}{2^3}+\frac{2}{3^3}+\frac{3}{4^3}+\frac{4}{5^3}+\frac{5}{6^3}+\cdots = \sum_{n=2}^{\infty}\frac{n-1}{n^3} = \sum_{n=2}^{\infty}\frac{1}{n^2}-\sum_{n=2}^{\infty}\frac{1}{n^3} = (p-1)-(q-1) = p-q.$$