SOLUTION FOR MARCH 2015

This is an approximate angle trisection method due to d'Ocagne. Given a small angle, θ, in a unit semicircle, let P be the midpoint of the segment AB and Q the midpoint of the arc CD. Show that angle $\alpha =$QPC $\approx \theta/3$. More precisely show that:

$$\lim_{\theta \to 0^+} \frac{\tan(\alpha)}{\theta} = \frac{1}{3}.$$

SOLUTION: Since the point D lies on the unit circle we can label its coordinates:

$$D = (\cos(\theta), \sin(\theta)).$$

Similarly:

$$Q = (\cos(\theta/2), \sin(\theta/2)).$$

Then by trigonometry:

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\theta/2)}{\frac{1}{2} + \cos(\theta/2)}.$$

Thus:

$$\frac{\tan(\alpha)}{\theta} = \frac{\sin(\theta/2)}{\theta} \frac{1}{\frac{1}{2} + \cos(\theta/2)}.$$

By L'Hopital's rule:

$$\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\theta} = \lim_{\theta \to 0} \frac{\frac{1}{2}\cos(\theta/2)}{1} = \frac{1}{2}$$

and also:

$$\lim_{\theta \to 0} \frac{1}{\frac{1}{2} + \cos(\theta/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

Therefore:

$$\lim_{\theta \to 0} \frac{\tan(\alpha)}{\theta} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$

One final remark: notice that as $\theta \to 0$ then $\alpha \to 0$ and again by L'Hopital's rule

$$\lim_{\theta \to 0} \frac{\tan(\alpha)}{\alpha} = 1$$

and so:

$$\lim_{\theta \to 0} \frac{\alpha}{\theta} = \lim_{\theta \to 0} \frac{\alpha}{\tan(\alpha)} \frac{\tan(\alpha)}{\theta} = 1 \cdot \frac{1}{3} = \frac{1}{3}.$$

Thus α is close to $\frac{1}{3}\theta$ when θ is small - i. e. α nearly trisects θ.