SOLUTION FOR OCTOBER 2016

a. Show that if A and B are linear transformations from $\mathbb{R}^N \to \mathbb{R}^N$ then it is impossible for:

$$AB - BA = I$$

where I is the identity map.

b. On the other hand show that there are linear transformations A, B (defined on some infinite dimensionsal space) so that:

$$AB - BA = I.$$

SOLUTION: In finite dimensions if $AB - BA = I$ then taking the trace of both sides gives

$$tr(AB - BA) = tr(I) = N.$$

Using the fact that $tr(AB) = tr(BA)$ and that $tr(C - D) = tr(C) - tr(D)$ we see that $tr(AB - BA) = tr(AB) - tr(BA) = 0$ and thus $0 = N$ which is impossible.

In infinite dimensions let us consider the vector space of polynomials with real coeeficients:

$$P = \{a_0 + a_1 x + \cdots + a_n x^n |\ a_0, a_1, \cdots, a_n \text{ are real}\}.$$

Now define $(Ap)(x) = p'(x)$ and $(Bp)(x) = xp(x)$. These are both linear and:

$((AB)(p))(x) = (A(xp))(x) = xp'(x) + p(x)$ by the product rule from calculus

and:

$$((BA)(p))(x) = (Bp')(x) = xp'(x).$$

Thus:

$$((AB - BA)(p))(x) = xp'(x) + p(x) - xp'(x) = p(x).$$

Therefore:

$$AB - BA = I \text{ on } P.$$